

DOUBLE-BOSONIZATION AND MAJID'S CONJECTURE, (IV): TYPE-CROSSINGS FROM A TO BCD

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ABSTRACT. Both in Majid's double-bosonization theory and in Rosso's quantum shuffle theory, the rank-recursive and type-crossing construction for $U_q(\mathfrak{g})$'s is still a remaining open question. Working in Majid's framework in this paper, based on the generalized double-bosonization Theorem we proved before, we further describe explicitly the type-crossing construction of $U_q(\mathfrak{g})$'s for $(BCD)_n$ series directly from type A_{n-1} via adding a pair of dual braided groups determined by a pair of (R, R') -matrices of type A derived from the respective suitably chosen representations. Combining with our results of the first three papers of this series, this solves Majid's conjecture, that is, any quantum group $U_q(\mathfrak{g})$ associated to a simple Lie algebra \mathfrak{g} can be grown out of $U_q(\mathfrak{sl}_2)$ recursively by a series of suitably chosen double-bosonization procedures.

1. INTRODUCTION AND OUR RESULTS

There are several well-known ways to understand the structure of quantum groups, for example, Drinfeld's quantum double of any finite-dimensional Hopf algebra introduced in [Dr1], the FRT-construction in [FRT] based on R -matrices limited to the classical types. Afterwards, Majid rediscovered the Radford biproduct, nowadays named "Radford-Majid bosonization", in the framework of a braided monoidal category of left modules over a Hopf algebra whose Drinfeld center is just the category of Yetter-Drinfeld modules over it (see [Ra], [RT], [M8], [M9], and for the first proof of the latter see that of Example 1.3 in [M10]), Majid then developed the double-bosonization theory, which improved the FRT-construction and extended Drinfeld's quantum double to generalized quantum double associated to a weakly quasitriangular dual pair (not necessarily nondegenerate) of braided

1991 *Mathematics Subject Classification.* Primary 16S40, 16W30, 17B37, 18D10; Secondary 17B10, 20G42, 81R50.

Key words and phrases. Double-bosonization, braided category, braided groups, type-crossing construction, normalized R -matrix, representations.

N. H., the corresponding author, supported by the NNSFC (Grant No. 11271131).

groups coming from the (co)module category of a (co)quasitriangular Hopf algebra. Another analogous construction in spirit was given by Sommerhäuser in [So] in a larger category of Yetter-Drinfeld modules, but the latter is seemly not suitable for the rank-recursive construction considered here.

Roughly speaking, the Majid's double-bosonization theory is given as follows. Assigned to a pair of dual braided groups B^\star, B covariant under a underlying quasitriangular Hopf algebra H , there is a new quantum group on the tensor space $B^\star \otimes H \otimes B$ by double-bosonization in [M1], consisting of H extended by B as additional 'positive roots' and its dual B^\star as additional 'negative roots'. On the other hand, Majid claimed that the rank-recursive construction of $U_q(\mathfrak{g})$'s can be obtained in principle in his context, and expected his double-bosonization allows to generate a tree of quantum groups. But how to do it? Except two examples in low rank given by Majid (see [M1, M6, M7]), it is still a remaining open question for almost 20 years with a main challenge from representation theory to elaborate the full tree structure of quantum groups generated by the double-bosonization procedures. The technical difficulties mainly involve in the exact choices of certain representations as well as treating with some nonstandard R -matrices derived from these representations. In general, as we known, it is difficult to capture some information encoded in a (nonstandard) R -matrix associated to some representation of $U_q(\mathfrak{g})$, for example, the spectral decomposition of R -matrix, to normalize R -matrices, etc. However, the authors have overcome some technical difficulties to give concretely the rank-recursive constructions of $U_q(\mathfrak{g})$'s for the $ABCD$ series in [HH1], the type-crossing constructions for types F_4, G_2 in [HH2] and the type-crossing rank-recursive constructions for types E_6, E_7, E_8 in [HH3]. Since Majid's framework on braided groups in certain braided categories has been developed currently into the framework of Nichols algebras in the Yetter-Drinfeld categories for classifying finite-dimensional pointed Hopf algebras due to Andruskiewitsch-Schneider and Heckenberger et al (see [AS1, AS2], [HS], etc.), we believe in some sense that the study on how the double-bosonization procedure yields what kinds of new quantum groups is significant. An interesting application of our constructions might be connected to a recent work of Cuntz and Lentner on Nichols algebras (see [CL]).

Some examples in low ranks given in [HH1] show that B_2, C_3, D_4 can be grown out of $U_q(\mathfrak{sl}_n)$, for $n = 2, 3, 4$, respectively. Then, what about the general type-crossing construction of the $(BCD)_n$ series directly from type A_{n-1} ? This is also a remaining problem of Rosso in his quantum shuffle setting (see [Ro]). If this problem is solved, we will confirm completely the Majid's conjecture on how step by step and directly from type A_1 to get all quantum groups $U_q(\mathfrak{g})$'s of the finite dimensional complex simple Lie algebras \mathfrak{g} . As a reminder, we mention that Rosso's quantum shuffle theory in [Ro] gives an intrinsic, or

functorial understanding of $U_q(\mathfrak{g})$ from the braidings. Rosso also asserted the rank-recursive or type-crossing constructions of all positive part $U_q(\mathfrak{n})$'s existing in his quantum shuffle setting and listed an interpreting recipe simply at the Lie theory level. However, for the completeness of both theories, it is desirable to know how to proceed them in the respective contexts. This also stimulates us to attack this question first in the Majid's framework.

The paper is organized as follows. In section 2, we recall some basic facts about the FRT-construction, the Majid's double-bosonization theorem and our generalized double-bosonization theorem ([HH2]). Section 3 is devoted to exploring the general type-crossing constructions for the $(BCD)_n$ series directly from node A_{n-1} . Firstly, based on the vector representation T_V of $U_q(\mathfrak{sl}_n)$ and the corresponding standard R -matrix, through choosing a matched matrix R' such that (R, R') derives a pair of dual braided groups $\tilde{V}^\vee(R), \tilde{V}(R)$ covariant under $U_q(\mathfrak{sl}_n)$, which is different from those used in [HH1], we construct the expected quantum group $U_q(\mathfrak{so}_{2n+1})$ of higher rank 1. Note that our construction procedure here for type B_n is a bit different from those of types C_n and D_n . Secondly, in order to construct the expected $U_q(\mathfrak{sp}_{2n})$ and $U_q(\mathfrak{so}_{2n})$ of higher rank 1, we consider the quantum symmetric square $\text{sym}^2 V$ and the quantum second exterior power $\wedge^2 V$ of the vector representation T_V of $U_q(\mathfrak{sl}_n)$. We first analyse the R -matrices R_{VV} corresponding to these representations, and obtain the minimal polynomials of PR_{VV} with our skillful techniques sufficiently exploiting the features of the representations we work with. We get a pair of matrices (R, R') satisfying $(PR+I)(PR'-I) = 0$, and new mutually dual braided groups $V^\vee(R', R_{21}^{-1}), V(R', R)$ covariant under $U_q(\mathfrak{sl}_n)$, and then to arrive at our required objects. As a final consequence of our results including those in [HH1, HH2, HH3], we confirm that Majid's conjecture is true, that is, starting with $U_q(\mathfrak{sl}_2)$, we obtain step by step all $U_q(\mathfrak{g})$'s associated with the finite-dimensional complex simple Lie algebras \mathfrak{g} by a series of delicately selected double-bosonization procedures. In section 4, we contrast Majid's double-bosonization with Rosso's quantum shuffle theory to deduce the same extended Dynkin diagrams of higher rank one, and draw the tree structure of quantum groups $U_q(\mathfrak{g})$'s, for all the finite-dimensional complex simple Lie algebras \mathfrak{g} , grown out of the source node A_1 inductively by a series of suitably chosen double-bosonization procedures we build.

2. PRELIMINARIES

In this paper, let k be the complex field, \mathbb{R} the real field, E the Euclidean space \mathbb{R}^n or a suitable subspace. Denote by ε_i the usual orthogonal unit vectors in \mathbb{R}^n . Let \mathfrak{g} be a finite-dimensional complex semisimple Lie algebra with simple roots α_i , λ_i the fundamental weight corresponding to simple root α_i . The Cartan matrix of \mathfrak{g} is (a_{ij}) , where $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$, and $d_i = \frac{(\alpha_i, \alpha_i)}{2}$. Let (H, \mathcal{R}) be a quasitriangular Hopf algebra, where $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$

is the universal R -matrix, $\mathcal{R}_{21} = \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)}$, denote by Δ, η, ϵ its coproduct, counit, unit, and by S its antipode. We shall use the Sweedler's notation: for $h \in H$, $\Delta(h) = h_1 \otimes h_2$. H^{op} (H^{cop}) denotes the opposite (co)algebra structure of H , respectively. \mathfrak{M}_H (${}_H\mathfrak{M}$) denotes the braided category consisting of right (left) H -modules. If there exists a coquasitriangular Hopf algebra A such that the dual pair (H, A) is a weakly quasitriangular, then \mathfrak{M}_H (${}_H\mathfrak{M}$) is equivalent to the braided category ${}^A\mathfrak{M}$ (\mathfrak{M}^A) consisting of left (right) A -comodules. For the detailed description of these, we left the readers to refer to Drinfeld's and Majid's papers [Dr2], [M2], [M3], and so on. By a braided group we mean a braided bialgebra or Hopf algebra in some braided category. In order to distinguish from the ordinary Hopf algebras, denote by $\underline{\Delta}, \underline{S}$ its coproduct and antipode, respectively.

2.1. FRT-construction. Let R be an invertible matrix obeying the quantum Yang-Baxter equation. There is a bialgebra $A(R)$ [FRT] corresponding to the R -matrix, called the FRT-bialgebra.

Definition 2.1. $A(R)$ is generated by 1 and $T = \{t_j^i\}$, having the following defining relations:

$$RT_1T_2 = T_2T_1R, \quad \Delta(T) = T \otimes T, \quad \epsilon(T) = I, \quad \text{where } T_1 = T \otimes I, \quad T_2 = I \otimes T.$$

$A(R)$ is a coquasitriangular bialgebra with $\mathcal{R} : A(R) \otimes A(R) \longrightarrow k$ such that $\mathcal{R}(t_j^i \otimes t_l^k) = R_{jl}^{ik}$, where R_{jl}^{ik} denotes the entry at row (ik) and column (jl) in the matrix R .

On the other hand, Δ in $A(R)$ induces the multiplication in $A(R)^* = \text{Hom}(A(R), k)$. In [FRT], U_R is defined to be the subalgebra of $A(R)^*$ generated by $L^\pm = (l_{ij}^\pm)$, with relations $(PRP)L_1^\pm L_2^\pm = L_2^\pm L_1^\pm (PRP)$, $(PRP)L_1^+ L_2^- = L_2^- L_1^+ (PRP)$, where P is the permutation matrix $P : P(u \otimes v) = v \otimes u$, l_{ij}^\pm is defined by $(l_{ij}^+, t_l^k) = R_{lj}^{ki}$, $(l_{ij}^-, t_l^k) = (R^{-1})_{lj}^{ik}$. Moreover, the algebra U_R is a bialgebra with coproduct $\Delta(L^\pm) = L^\pm \otimes L^\pm$, and counit $\epsilon(l_{ij}^\pm) = \delta_{ij}$ (also cf. [KS]).

Specially, when R is one of the classical R -matrices, bialgebra $A(R)$ has a quotient co-quasitriangular Hopf algebra, denoted by $\text{Fun}(G_q)$ or $O_q(G)$, and then U_R has a corresponding quotient quasitriangular Hopf algebra, which is isomorphic to the extended quantized enveloping algebra $U_q^{\text{ext}}(\mathfrak{g})$ (with groups-like elements indexed in a refined weight lattice in comparison with $U_q(\mathfrak{g})$). Moreover, there exists a (non-degenerate) dual pairing \langle, \rangle between $O_q(G)$ and $U_q^{\text{ext}}(\mathfrak{g})$. The way of getting the resulting quasitriangular algebras $U_q^{\text{ext}}(\mathfrak{g})$ is the so-called FRT-construction of the quantized enveloping algebras (for the classical types).

Remark 2.1. The R -matrices used in Majid's paper [M1] are a bit different from those as in [FRT], which are the conjugations $P \circ \cdot \circ P$ of the ordinary R -matrices. It is easy to check that $A(P \circ R \circ P) = A(R)^{op}$, where $(P \circ R \circ P)_{kl}^{ij} = R_{lk}^{ji}$. Note that we will use the Majid's notation for R -matrices as in [M1] in the remaining sections of this paper.

2.2. Majid's double-bosonization. Majid [M1] proposed the concept of a weakly quasi-triangular dual pair via his insight on more examples on matched pairs of bialgebras or Hopf algebras in [M5]. This allowed him to establish a theory of double-bosonization in a broad framework that generalized the FRT's construction which was limited to the classical types.

Definition 2.2. Let (H, A) be a pair of Hopf algebras equipped with a dual pairing $\langle \cdot, \cdot \rangle$ and convolution-invertible algebra/anti-coalgebra maps $\mathcal{R}, \bar{\mathcal{R}} : A \rightarrow H$ obeying

$$\langle \bar{\mathcal{R}}(a), b \rangle = \langle \mathcal{R}^{-1}(b), a \rangle, \quad \partial^R h = \mathcal{R} * (\partial^L h) * \mathcal{R}^{-1}, \quad \partial^R h = \bar{\mathcal{R}} * (\partial^L h) * \bar{\mathcal{R}}^{-1}$$

for $a, b \in A, h \in H$. Here $*$ is the convolution product in $\text{hom}(A, H)$ and $(\partial^L h)(a) = \langle h_{(1)}, a \rangle h_{(2)}$, $(\partial^R h)(a) = h_{(1)} \langle h_{(2)}, a \rangle$ are left, right “differentiation operators” regarded as maps $A \rightarrow H$ for fixed h .

For any quasitriangular Hopf algebra (H, \mathcal{R}) (where $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$), Majid [M8] rediscovered a special form of the Radford biproduct in the braided category ${}_H\mathfrak{M}$ of left H -modules, currently named the “Radford-Majid bosonization”. Let us recall the definition.

Definition 2.3. Let B be a braided group in ${}_H\mathfrak{M}$, then its bosonization is the Hopf algebra $B \bowtie H$ defined as $B \otimes H$ (as a vector space) with the product, coproduct and antipode

$$\begin{aligned} (b \otimes h)(c \otimes g) &= b(h_{(1)} \triangleright c) \otimes h_{(2)}g, \\ \Delta(b \otimes h) &= \underline{b}_{(1)} \otimes \mathcal{R}^{(2)}h_{(1)} \otimes \mathcal{R}^{(1)} \triangleright \underline{b}_{(2)} \otimes h_{(2)}, \\ S(b \otimes h) &= (Sh_{(2)})u\mathcal{R}^{(1)} \triangleright \underline{S}b \otimes S(\mathcal{R}^{(2)}h_{(1)}); \quad u \equiv (S\mathcal{R}^{(2)})\mathcal{R}^{(1)}. \end{aligned}$$

On the other hand, the right-handed version for $B \in \mathfrak{M}_H$ is $H \ltimes B$ defined by

$$\begin{aligned} (h \otimes b)(g \otimes c) &= hg_{(1)} \otimes (b \triangleleft g_{(2)})c, \\ \Delta(h \otimes b) &= h_{(1)} \otimes \underline{b}_{(1)} \triangleleft \mathcal{R}^{(1)} \otimes h_{(2)}\mathcal{R}^{(2)} \otimes \underline{b}_{(2)}, \\ S(h \otimes b) &= S(h_{(2)}\mathcal{R}^{(2)}) \otimes \underline{S}b \triangleleft \mathcal{R}^{(1)}vSh_{(1)}; \quad v \equiv \mathcal{R}^{(1)}(S\mathcal{R}^{(2)}). \end{aligned}$$

The corresponding formulae for bosonizations in the comodule categories $\mathfrak{M}^A, {}^A\mathfrak{M}$, respectively are trivially obtained by the usual conversions of the module formulae.

Let C, B be a pair of braided groups in \mathfrak{M}_H , which are called dually paired if there is an intertwiner $ev : C \otimes B \rightarrow k$ such that $ev(cd, b) = ev(d, \underline{b}_{(1)})ev(c, \underline{b}_{(2)})$, $ev(c, ab) = ev(\underline{c}_{(2)}, a)ev(\underline{c}_{(1)}, b)$, $\forall a, b \in B, c, d \in C$. Then $C^{op/cop}$ (with opposite product and coproduct) is a Hopf algebra in ${}_H\mathfrak{M}$, which is dual to B in the sense of an ordinary dual pairing $\langle \cdot, \cdot \rangle$, which is H -bicovariant: $\langle h \triangleright c, b \rangle = \langle c, b \triangleleft h \rangle$ for all $h \in H$. Let $\bar{C} = (C^{op/cop})^{cop}$, then \bar{C} is a braided group in $\bar{H}\mathfrak{M}$, where \bar{H} is $(H, \mathcal{R}_{21}^{-1})$. So there are two Hopf algebras $H \bowtie B$

(bosonization) and $\bar{C} \rtimes H$ (bosonization) by the above Definition. What relations do there exist between them? Majid gave the following double-bosonization theorem.

Theorem 2.1. (Majid) *On the tensor space $\bar{C} \otimes H \otimes B$, there is a unique Hopf algebra structure $U = U(\bar{C}, H, B)$ such that $H \rtimes B$ (bosonization) and $\bar{C} \rtimes H$ (bosonization) are sub-Hopf algebras by the canonical inclusions and*

$$bc = (\mathcal{R}_1^{(2)} \triangleright c_{(2)}) \mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)} (b_{(2)} \triangleleft \mathcal{R}_2^{-(1)}) \langle \mathcal{R}_1^{(1)} \triangleright c_{(1)}, b_{(1)} \triangleleft \mathcal{R}_2^{(1)} \rangle \cdot \\ \cdot \langle \mathcal{R}_1^{-(2)} \triangleright \bar{S} c_{(3)}, b_{(3)} \triangleleft \mathcal{R}_2^{-(2)} \rangle,$$

for all $b \in B$, $c \in \bar{C}$ viewed in U . Here $\mathcal{R}_1, \mathcal{R}_2$ are distinct copies of the quasitriangular structure \mathcal{R} of H . The product, coproduct of U are given by

$$(c \otimes h \otimes b) \cdot (d \otimes g \otimes a) = c(h_{(1)} \mathcal{R}_1^{(2)} \triangleright d_{(2)}) \otimes h_{(2)} \mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)} g_{(1)} \otimes (b_{(2)} \triangleleft \mathcal{R}_2^{-(1)} g_{(2)}) a \\ \langle \mathcal{R}_1^{(1)} \triangleright d_{(1)}, b_{(1)} \triangleleft \mathcal{R}_2^{(1)} \rangle \langle \mathcal{R}_1^{-(2)} \triangleright \bar{S} d_{(3)}, b_{(3)} \triangleleft \mathcal{R}_2^{-(2)} \rangle;$$

$$\Delta(c \otimes h \otimes b) = c_{(1)} \otimes \mathcal{R}^{-(1)} h_{(1)} \otimes b_{(1)} \triangleleft \mathcal{R}^{(1)} \otimes \mathcal{R}^{-(2)} \triangleright c_{(2)} \otimes h_{(2)} \mathcal{R}^{(2)} \otimes b_{(2)}.$$

Moreover, the antipodes of $H \rtimes B$ and $\bar{C} \rtimes H$ can be extended to an antipode $S : U \rightarrow U$ by the two extensions $S(chb) = (Sb) \cdot (S(ch))$ and $S(chb) = (S(hb)) \cdot (Sc)$.

Remark 2.2. *If there exists a coquasitriangular Hopf algebra A such that (H, A) is a weakly quasitriangular dual pair, and b, c are primitive elements, then some relations simplify to*

$$[b, c] = \mathcal{R}(b^{(1)}) \langle c, b^{(2)} \rangle - \langle c^{(1)}, b \rangle \bar{\mathcal{R}}(c^{(2)}); \quad (C2)$$

$$\Delta b = b^{(2)} \otimes \mathcal{R}(b^{(1)}) + 1 \otimes b, \quad \Delta c = c \otimes 1 + \bar{\mathcal{R}}(c^{(2)}) \otimes c^{(1)}. \quad (C3)$$

To each R , Majid associated two braided groups $V(R', R)$, $V^\vee(R', R_{21}^{-1})$ in the braided categories ${}^{A(R)}\mathfrak{M}$, $\mathfrak{M}^{A(R)}$, called the braided vector algebra, the braided covector algebra, respectively in [M4] (see below).

Proposition 2.1. *Suppose that R' is another matrix such that (i) $R_{12}R_{13}R'_{23} = R'_{23}R_{13}R_{12}$, $R_{23}R_{13}R'_{12} = R'_{12}R_{13}R_{23}$, (ii) $(PR + 1)(PR' - 1) = 0$, (iii) $R_{21}R'_{12} = R'_{21}R_{12}$, where P is a permutation matrix with the entry $P_{kl}^{ij} = \delta_{il}\delta_{jk}$. Then the braided-vector algebra $V(R', R)$ defined by generators $1, \{e^i \mid i = 1, \dots, n\}$, and relations*

$$e^i e^j = \sum_{a,b} R'^{ji}_{ab} e^a e^b$$

forms a braided group with

$$\underline{\Delta}(e^i) = e^i \otimes 1 + 1 \otimes e^i, \quad \underline{\epsilon}(e^i) = 0, \quad \underline{S}(e^i) = -e^i, \quad \Psi(e^i \otimes e^j) = \sum_{a,b} R_{ab}^{ji} e^a \otimes e^b$$

in braided category ${}^{A(R)}\mathfrak{M}$.

Under the duality $\langle f_j, e^i \rangle = \delta_{ij}$, the braided-covector algebra $V^\vee(R', R_{21}^{-1})$ defined by 1 and $\{f_j \mid j = 1, \dots, n\}$, and relations

$$f_i f_j = \sum_{a,b} f_b f_a R'_{ij}{}^{ab}$$

forms another braided group with

$$\underline{\Delta}(f_i) = f_i \otimes 1 + 1 \otimes f_i, \quad \underline{\epsilon}(f_i) = 0, \quad \underline{S}(f_i) = -f_i, \quad \Psi(f_i \otimes f_j) = \sum_{a,b} f_b \otimes f_a R_{ij}^{ab}$$

in braided category $\mathfrak{M}^{A(R)}$.

Remark 2.3. PR obeys some minimal polynomial equation $\prod_i (PR - x_i) = 0$. For each nonzero eigenvalue x_i , we can normalize R so that $x_i = -1$. Then set $R' = P + P \prod_{j \neq i} (PR - x_j)$. This satisfies the conditions (i) – (iii), which gives us at least one mutually dual pair of braided (co)vectors for each nonzero eigenvalue of PR .

2.3. Generalized double-bosonization Theorem. Majid obtained another bialgebra $\widetilde{U(R)}$ by the double cross product bialgebra of $A(R)^{op}$ in [M5], generated by m^\pm with

$$Rm_1^\pm m_2^\pm = m_2^\pm m_1^\pm R, \quad Rm_1^+ m_2^- = m_2^- m_1^+ R, \quad \Delta((m^\pm)_j^i) = (m^\pm)_j^a \otimes (m^\pm)_a^i, \quad \epsilon((m^\pm)_j^i) = \delta_{ij},$$

where $(m^\pm)_j^i$ denotes the entry at row i and column j in m^\pm . Moreover, $(\widetilde{U(R)}, A(R))$ is a weakly quasitriangular dual pair with

$$\langle (m^+)_j^i, t_l^k \rangle = R_{jl}^{ik}, \quad \langle (m^-)_j^i, t_l^k \rangle = (R^{-1})_{lj}^{ki}, \quad \mathcal{R}(T) = m^+, \quad \bar{\mathcal{R}}(T) = m^-.$$

When the R -matrix R_{VV} we work with is nonstandard (By “standard” we mean that the R -matrix is obtained by the vector representation of $U_q(\mathfrak{g})$ when \mathfrak{g} is limited to $ABCD$ series) (see section 3), we obtained the following

Theorem 2.2. [HH2] When the R -matrix R_{VV} is nonstandard, the above weakly quasitriangular dual pair can be descended to a mutually dual pair of Hopf algebras $(U_q^{ext}(\mathfrak{g}), H_{\lambda R})$ with the correct modification:

$$\langle (m^+)_j^i, t_l^k \rangle = R_{VV}^{ik} = \lambda R_{jl}^{ik}, \quad \langle (m^-)_j^i, t_l^k \rangle = R_{VV}^{-1}{}^{ki} = \lambda^{-1} (R^{-1})_{lj}^{ki}.$$

$$\langle (m^+)_j^i, \tilde{t}_l^k \rangle = (R_{VV}^{t_2})^{-1}{}^{ik} = ((\lambda R)^{t_2})^{-1}{}^{ik}, \quad \langle (m^-)_j^i, \tilde{t}_l^k \rangle = [(R_{VV}^{-1})^{t_1}]^{-1}{}^{lj} = [(\lambda^{-1} R^{-1})^{t_1}]^{-1}{}^{lj}{}_{ki}.$$

Such λ is called a normalization constant of quantum groups. The convolution-invertible algebra/anti-coalgebra maps $\mathcal{R}, \bar{\mathcal{R}}$ in $\text{hom}(H_{\lambda R}, U_q^{ext}(\mathfrak{g}))$ are

$$\mathcal{R}(t_j^i) = (m^+)_j^i, \quad \mathcal{R}(\tilde{t}_j^i) = (m^+)^{-1}{}_{ij}, \quad \bar{\mathcal{R}}(t_j^i) = (m^-)_j^i, \quad \bar{\mathcal{R}}(\tilde{t}_j^i) = (m^-)^{-1}{}_{ij},$$

where $U_q^{\text{ext}}(\mathfrak{g})$ is the FRT-form of $U_q(\mathfrak{g})$ or extended quantized enveloping algebra adjoined by the elements $K_i^{\frac{1}{m}}$, $H_{\lambda R}$ is the Hopf algebra associated with $A(R_{VV})$, generated by t_j^i, \tilde{t}_j^i and subject to certain relations.

With the weakly quasitriangular dual pair of Hopf algebras in the above Theorem, we obtain that the braided (co)vector algebras $V(R', R) \in {}^{H_{\lambda R}}\mathfrak{M}$ and $V^\vee(R', R_{21}^{-1}) \in \mathfrak{M}^{H_{\lambda R}}$. However, in order to yield the required quantum group of higher rank 1, we need centrally extend the pair $(U_q^{\text{ext}}(\mathfrak{g}), H_{\lambda R})$ to the pair

$$\left(\widetilde{U_q^{\text{ext}}(\mathfrak{g})} = U_q^{\text{ext}}(\mathfrak{g}) \otimes k[c, c^{-1}], \widetilde{H_{\lambda R}} = H_{\lambda R} \otimes k[g, g^{-1}] \right)$$

with the action $e^i \triangleleft c = \lambda e^i$, $f_i \triangleleft c = \lambda f_i$ and the extended pair $\langle c, g \rangle = \lambda$. In this way, we easily see that the braided algebras $V(R', R) \in {}^{\widetilde{H_{\lambda R}}}\mathfrak{M}$ and $V^\vee(R', R_{21}^{-1}) \in \mathfrak{M}^{\widetilde{H_{\lambda R}}}$.

With these, we have the following *generalized double-bosonization Theorem* in [HH2] by Theorem 2.1.

Theorem 2.3. *Let R_{VV} be the R -matrix corresponding to the irreducible representation T_V of $U_q(\mathfrak{g})$. There exists a normalization constant λ such that $\lambda R = R_{VV}$. Then the new quantum group $U = U(V^\vee(R', R_{21}^{-1}), \widetilde{U_q^{\text{ext}}(\mathfrak{g})}, V(R', R))$ has the following the cross relations:*

$$\begin{aligned} c f_i &= \lambda f_i c, \quad e^i c = \lambda c e^i, \quad [c, m^\pm] = 0, \quad [e^i, f_j] = \delta_{ij} \frac{(m^+)_j^i c^{-1} - c (m^-)_j^i}{q_* - q_*^{-1}}, \\ e^i (m^+)_k^j &= R_{VV}^{ji} (m^+)_k^a e^b, \quad (m^-)_j^i e^k = R_{VV}^{ki} e^a (m^-)_j^b, \\ (m^+)_j^i f_k &= f_b (m^+)_a^i R_{VV}^{ab}{}_{jk}, \quad f_i (m^-)_k^j = (m^-)_b^j f_a R_{VV}^{ab}{}_{ik}, \end{aligned}$$

and the coproduct:

$$\Delta c = c \otimes c, \quad \Delta e^i = e^a \otimes (m^+)_a^i c^{-1} + 1 \otimes e^i, \quad \Delta f_i = f_i \otimes 1 + c (m^-)_i^a \otimes f_a,$$

and the counit $\epsilon e^i = \epsilon f_i = 0$.

We can normalize e^i such that the factor $q_* - q_*^{-1}$ in Theorem 2.3 meets the situation we need. On the other hand, the FRT-generators $(m^\pm)_j^i$ can be obtained by the following lemma, which was proved in [HH2].

Lemma 2.1. *The elements $l_{ij}^\pm \in U_q(\mathfrak{g})$ are the L -functionals associated with some representation for $U_q(\mathfrak{g})$. Then the FRT-generators $(m^\pm)_j^i$ can be obtained by $(m^\pm)_j^i = S(l_{ij}^\pm)$, where S is the antipode.*

3. TYPE-CROSSING CONSTRUCTION OF QUANTUM GROUPS FOR BCD SERIES

Besides the results in [HH1, HH2, HH3], the remaining cases for the type-crossing construction of $U_q(\mathfrak{g})$'s needed to be considered are the $(BCD)_n$ series directly starting from type A_{n-1} . The authors will consider them for the first time in this section. The above R -matrix datum R_{VV} is given by the universal R -matrix of $U_h(\mathfrak{g})$ and a certain representation T_V . Set $\mathfrak{R} = \sum_{r_1, \dots, r_n=0}^{\infty} \prod_{j=1}^n \frac{(1-q_{\beta_j}^{-2})^{r_j}}{[r_j]_{q_{\beta_j}}!} q_{\beta_j}^{\frac{r_j(r_j+1)}{2}} E_{\beta_j}^{r_j} \otimes F_{\beta_j}^{r_j}$ (see [KS]). Then $R_{VV} = B_{VV} \circ (T_V \otimes T_V)(\mathfrak{R})$, where B_{VV} denotes the linear operator on $V \otimes V$ given by $B_{VV}(v \otimes w) := q^{(\mu, \mu')} v \otimes w$, for $v \in V_{\mu}, w \in V_{\mu'}$.

In the sequel, we will take a basis of V with weight-raising indices such that operators $T_V(E_i)$'s and $T_V(F_i)$'s raise and lower the indices of the basis respectively, then the R -matrix R_{VV} is upper triangular with respect to the lexicographic order on the basis of $V^{\otimes 2}$ induced by the chosen basis of V .

3.1. $A_{n-1} \implies B_n$. When $\mathfrak{g} = A_{n-1}$, the length of each simple root is $(\alpha_i, \alpha_i) = 2, 1 \leq i \leq n-1$, and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, the corresponding i -th fundamental weight is

$$\lambda_i = \frac{1}{n} [(n-i)\alpha_1 + 2(n-i)\alpha_2 + \dots + (i-1)(n-i)\alpha_{i-1} + i(n-i)\alpha_i + i(n-i-1)\alpha_{i+1} + \dots + i\alpha_{n-1}].$$

The standard R -matrix for type A corresponding to λ_1 satisfies the quadratic equation $(PR - qI)(PR + q^{-1}I) = 0$ in [FRT, KS], where (here we follow the convention in Remark 2.1)

$$R_{kl}^{ij} = q^{\delta_{ij}} \delta_{ik} \delta_{jl} + (q - q^{-1}) \delta_{il} \delta_{jk} \theta(j-i), \quad \text{where } \theta(k) = \begin{cases} 1 & k > 0, \\ 0 & k \leq 0. \end{cases} \quad (\star)$$

Moreover, the weakly quasitriangular dual pair $(U_q^{ext}(\mathfrak{sl}_n), O_q(SL(n)))$ is given by

$$\langle (m^+)_j, t_l^k \rangle = \lambda R_{jl}^{ik}, \quad \langle (m^-)_j, t_l^k \rangle = \lambda^{-1} (R^{-1})_{lj}^{ki},$$

where $\lambda = q^{-\frac{1}{n}}$. Specially, if we choose $R' = P$, then $(PR + I)(PR' - I) = 0$ for any R . But when $R' = P$, the quadratic relations in Proposition 2.1 are $e^i e^j = \sum_{a,b} R'_{ab}{}^{ji} e^a e^b = e^i e^j$, and $f_i f_j = \sum_{a,b} f_b f_a R'_{ij}{}^{ab} = f_i f_j$. Namely, the resulting braided (co)vector algebras $V^\vee(P, R_{21}^{-1}), V(P, R)$ are the free braided (co)vector algebras $V^\vee(R), V(R)$. However, in what follows, we will find that the pair between them induced from $\langle f_j, e^i \rangle = \delta_{ij}$ is degenerate.

Proposition 3.1. *For the standard R -matrix (\star) for type A , the q -Serre-like relations below belong to the radicals of dual pair.*

$$(1) \quad \langle f, (e^i)^2 e^j + q e^j (e^i)^2 - (1+q) e^i e^j e^i \rangle = 0 \quad (i > j), \quad \text{for any } f \in V^\vee(R),$$

$$(2) \quad \langle f_j(f_i)^2 + q^{-1}(f_i)^2 f_j - (1 + q^{-1})f_i f_j f_i, e \rangle = 0 \quad (i > j), \quad \text{for any } e \in V(R).$$

Proof. The braiding Ψ satisfies $\Psi_{V, V \otimes V} = (id \otimes \Psi_{V, V})(\Psi_{V, V} \otimes id)$, $\Psi_{V \otimes V, V} = (\Psi_{V, V} \otimes id)(id \otimes \Psi_{V, V})$. The structure of braided algebras in braided (co)vector algebras $V^\vee(R)$, $V(R)$ are

$$\Psi_{V, V} \circ (id \otimes \cdot) = (\cdot \otimes id) \circ \Psi_{V, V \otimes V}, \quad \Psi_{V, V} \circ (\cdot \otimes id) = (id \otimes \cdot) \circ \Psi_{V \otimes V, V}.$$

With the value of every entry in equality (\star) , we obtain the following braiding

$$\Psi(e^k \otimes e^k) = qe^k \otimes e^k, \text{ for any } k, \quad \Psi(e^m \otimes e^n) = \begin{cases} e^n \otimes e^m + (q - q^{-1})e^m \otimes e^n, & m > n; \\ e^n \otimes e^m, & m < n. \end{cases}$$

Then we have

$$\begin{aligned} \underline{\Delta}(e^k e^k) &= 1 \otimes e^k e^k + (1 + q)e^k \otimes e^k + e^k e^k \otimes 1, \quad \text{for any } k, \\ \underline{\Delta}(e^m e^n) &= \begin{cases} 1 \otimes e^m e^n + (1 + q - q^{-1})e^m \otimes e^n + e^n \otimes e^m + e^m e^n \otimes 1, & m > n; \\ 1 \otimes e^m e^n + e^m \otimes e^n + e^n \otimes e^m + e^m e^n \otimes 1, & m < n. \end{cases} \end{aligned}$$

Then we have $\langle f_k f_k, e^k e^k \rangle = 1 + q$, and

$$\langle f_m f_n, e^m e^n \rangle = \begin{cases} 1 + q - q^{-1}, & m > n, \\ 1, & m < n, \end{cases} \quad \langle f_n f_m, e^m e^n \rangle = \begin{cases} 1, & m > n, \\ 1, & m < n. \end{cases}$$

Firstly, we will consider these elements in the quadratic homogeneous space, which are some q -relations $a_1(e^i)^2 + a_2(e^j)^2 + a_3e^i e^j + a_4e^j e^i$ ($i > j$), where $a_i \in k[q, q^{-1}]$. We only need to consider the value $\langle f_a f_b, a_1(e^i)^2 + a_2(e^j)^2 + a_3e^i e^j + a_4e^j e^i \rangle$ for any a, b . According to $\langle cd, b \rangle = \langle c, b_{(1)} \rangle \langle d, b_{(2)} \rangle$, $\langle c, ab \rangle = \langle c_{(1)}, a \rangle \langle c_{(2)}, b \rangle$, and $\underline{\epsilon}(e^i) = 0$, $\underline{\epsilon}(f_i) = 0$. If the elements $a_1(e^i)^2 + a_2(e^j)^2 + a_3e^i e^j + a_4e^j e^i$ belong to the right radical, then for any a, b , we obtain

$$\begin{aligned} \langle f_a f_b, a_1(e^i)^2 + a_2(e^j)^2 + a_3e^i e^j + a_4e^j e^i \rangle \\ = a_1(1 + q)\langle f_a, e^i \rangle \langle f_b, e^i \rangle + a_2(1 + q)\langle f_a, e^j \rangle \langle f_b, e^j \rangle \\ + a_3(1 + q - q^{-1})\langle f_a, e^i \rangle \langle f_b, e^j \rangle + a_3\langle f_a, e^j \rangle \langle f_b, e^i \rangle \\ + a_4\langle f_a, e^j \rangle \langle f_b, e^i \rangle + a_4\langle f_a, e^i \rangle \langle f_b, e^j \rangle = 0. \end{aligned}$$

We only need to consider the situations $f_a f_b = f_i f_i, f_j f_j, f_i f_j, f_j f_i$, respectively, and get the following equations

$$\begin{cases} a_1(1 + q) = 0, \\ a_2(1 + q) = 0, \\ a_3(1 + q - q^{-1}) + a_4 = 0, \\ a_3 + a_4 = 0. \end{cases}$$

Solving it, we obtain $a_1 = a_2 = a_3 = a_4 = 0$. So the dual pair in the quadratic homogeneous space is non-degenerate. Then we will consider the cubic homogeneous space consisting of the elements $b_1(e^i)^3 + b_2(e^j)^3 + b_3(e^i)^2 e^j + b_4e^i e^j e^i + b_5e^j(e^i)^2$.

$$\underline{\Delta}((e^k)^3) = 1 \otimes (e^k)^3 + (e^k)^3 \otimes 1 + (1 + q + q^2)e^k \otimes (e^k)^2 + (1 + q + q^2)(e^k)^2 \otimes e^k, \quad \text{for any } k.$$

$$\underline{\Delta}((e^i)^2 e^j) = \begin{cases} 1 \otimes (e^i)^2 e^j + (q + q^2 - q^{-1})(e^i)^2 \otimes e^j + (q + q^2)e^i \otimes e^i e^j \\ + e^j \otimes (e^i)^2 + (q - q^{-1})e^i \otimes e^j e^i + (1 + q)e^i e^j \otimes e^i + (e^i)^2 e^j \otimes 1 & i > j; \\ 1 \otimes (e^i)^2 e^j + (e^i)^2 \otimes e^j + (1 + q)e^i \otimes e^i e^j \\ + e^j \otimes (e^i)^2 + (1 + q)e^i e^j \otimes e^i + (e^i)^2 e^j \otimes 1 & i < j. \end{cases}$$

With a similar method, we get

$$\underline{\Delta}(e^i e^j e^i) = \begin{cases} 1 \otimes e^i e^j e^i + e^i e^j e^i \otimes 1 + q e^j e^i \otimes e^i + (1 + q - q^{-1})(e^i)^2 \otimes e^j \\ + q e^i \otimes e^i e^j + e^j \otimes (e^i)^2 + (1 + q - q^{-1})e^i \otimes e^j e^i + e^i e^j \otimes e^i, & i > j; \\ 1 \otimes e^i e^j e^i + e^i e^j e^i \otimes 1 + (1 + q - q^{-1})e^j \otimes (e^i)^2 \\ + (1 + q - q^{-1})e^i e^j \otimes e^i + q e^i \otimes e^i e^j + (e^i)^2 \otimes e^j + q e^j e^i \otimes e^i, & i < j. \end{cases}$$

$$\underline{\Delta}(e^j (e^i)^2) = \begin{cases} 1 \otimes e^j (e^i)^2 + e^j (e^i)^2 \otimes 1 + (e^i)^2 \otimes e^j + (1 + q)e^i \otimes e^j e^i \\ + e^j \otimes (e^i)^2 + (1 + q)e^j e^i \otimes e^i, & i > j; \\ 1 \otimes e^j (e^i)^2 + e^j (e^i)^2 \otimes 1 + (1 + q)e^i \otimes e^j e^i + (e^i)^2 \otimes e^j \\ + (q + q^2 - q^{-1})e^j \otimes (e^i)^2 + (q^2 - 1)e^j e^i \otimes e^i + (q - q^{-1})e^i e^j \otimes e^i, & i < j. \end{cases}$$

If $b_1(e^i)^3 + b_2(e^j)^3 + b_3(e^i)^2 e^j + b_4 e^i e^j e^i + b_5 e^j (e^i)^2$ belong to the right radical, then for any a, b, c , when $i < j$,

$$\begin{aligned} & \langle f_a f_b f_c, b_1(e^i)^3 + b_2(e^j)^3 + b_3(e^i)^2 e^j + b_4 e^i e^j e^i + b_5 e^j (e^i)^2 \rangle \\ &= (1 + q + q^2)b_1 \langle f_a f_b, e^i e^i \rangle \langle f_c, e^i \rangle + (1 + q + q^2)b_2 \langle f_a f_b, e^j e^j \rangle \langle f_c, e^j \rangle \\ &+ (b_3 + b_4 + b_5) \langle f_a f_b, e^i e^i \rangle \langle f_c, e^j \rangle + [q b_4 + (q^2 - 1)b_5] \langle f_a f_b, e^i e^j \rangle \langle f_c, e^i \rangle \\ &+ [(1 + q)b_3 + (1 + q - q^{-1})b_4 + (q - q^{-1})b_5] \langle f_a f_b, e^i e^j \rangle \langle f_c, e^i \rangle = 0. \end{aligned}$$

Then we only need to consider the situations $f_a f_b f_c = f_i f_i f_i, f_j f_j f_j, f_i f_i f_j, f_j f_i f_i, f_i f_j f_i$, respectively, we obtain the following equations

$$\begin{cases} (1 + q + q^2)(1 + q)b_1 = 0, \\ (1 + q + q^2)(1 + q)b_2 = 0, \\ (1 + q)(b_3 + b_4 + b_5) = 0, \\ (1 + q - q^{-1})[q b_4 + (q^2 - 1)b_5] + (1 + q)b_3 + (1 + q - q^{-1})b_4 + (q - q^{-1})b_5 = 0, \\ q b_4 + (q^2 - 1)b_5 + (1 + q)b_3 + (1 + q - q^{-1})b_4 + (q - q^{-1})b_5 = 0. \end{cases}$$

Solving it, we obtain $b_1 = b_2 = b_3 = b_4 = b_5 = 0$. We will consider the situation $i > j$.

$$\begin{aligned} & \langle f_a f_b f_c, b_1(e^i)^3 + b_2(e^j)^3 + b_3(e^i)^2 e^j + b_4 e^i e^j e^i + b_5 e^j (e^i)^2 \rangle \\ &= (1 + q + q^2)b_1 \langle f_a f_b, e^i e^i \rangle \langle f_c, e^i \rangle + (1 + q + q^2)b_2 \langle f_a f_b, e^j e^j \rangle \langle f_c, e^j \rangle \\ &+ [(q + q^2 - q^{-1})b_3 + (1 + q - q^{-1})b_4 + b_5] \langle f_a f_b, e^i e^i \rangle \langle f_c, e^j \rangle \\ &+ [q b_4 + (1 + q)b_5] \langle f_a f_b, e^j e^j \rangle \langle f_c, e^i \rangle + [(1 + q)b_3 + b_4] \langle f_a f_b, e^i e^j \rangle \langle f_c, e^i \rangle. \end{aligned}$$

We also only need to consider the situations $f_a f_b f_c = f_i f_i f_i, f_j f_j f_j, f_i f_i f_j, f_j f_j f_i, f_i f_j f_i$, respectively, and obtain the following equations

$$\begin{cases} (1+q+q^2)(1+q)b_1 = 0, \\ (1+q+q^2)(1+q)b_2 = 0, \\ (1+q)[(q+q^2-q^{-1})b_3 + (1+q-q^{-1})b_4 + b_5] = 0, \\ (1+q-q^{-1})[(1+q)b_3 + b_4] + [qb_4 + (1+q)b_5] = 0, \\ (1+q)b_3 + b_4 + qb_4 + (1+q)b_5 = 0. \end{cases}$$

Solving it, we get $b_1 = b_2 = 0, b_4 = -(1+q)b_3, b_5 = qb_3$. So only the elements $b_3[(e^i)^2 e^j + qe^j(e^i)^2 - (1+q)e^i e^j e^i]$ ($i > j$) belong to the right radical in the cubic homogenous space of $V(R)$. With a similar analysis, we see that only the elements $f_j(f_i)^2 + q^{-1}(f_i)^2 f_j - (1+q^{-1})f_i f_j f_i$ ($i > j$) belong to the left radical in the cubic homogenous space of $V^\vee(R)$. \square

With the above Proposition 3.1, we pass to the quotients by the Hopf ideals generated by $(e^i)^2 e^j + qe^j(e^i)^2 - (1+q)e^i e^j e^i$ and $f_j(f_i)^2 + q^{-1}(f_i)^2 f_j - (1+q^{-1})f_i f_j f_i$ ($i > j$), respectively. Denote by $\tilde{V}^\vee(R), \tilde{V}(R)$ the respective quotient braided (co)vector algebras. With these, by the generalized double-bosonization Theorem (Theorem 2.3), we obtain the following

Theorem 3.1. *With $\lambda = q^{-\frac{1}{n}}$, identify $e^n, f_n, (m^+)_n^n c^{-1}$ with the additional simple root vectors E_n, F_n and the group-like element K_n . Then the resulting new quantum group $U(\tilde{V}^\vee(R), U_q^{\text{ext}}(\mathfrak{sl}_n), \tilde{V}(R))$ is exactly the $U_q(\mathfrak{so}_{2n+1})$ with $K_i^{\pm \frac{1}{n}}$ adjoined.*

Proof. Corresponding to the vector representation of $U_q(\mathfrak{sl}_n)$, all the diagonal and minor diagonal entries we need in the matrices m^\pm have been given in [HH1]. For the identification in Theorem 3.1, it easily follows from Theorem 2.3 that

$$[E_n, F_n] = \frac{K_n - K_n^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad \Delta(E_n) = E_n \otimes K_n + 1 \otimes E_n, \quad \Delta(F_n) = F_n \otimes 1 + K_n^{-1} \otimes F_n.$$

We just demonstrate how to explore the cross relations between new simple root vectors E_n, F_n and group-like element K_n , and the q -Serre relations between E_n (F_n) and E_{n-1} (F_{n-1}). Other relations can be obtained by Theorem 2.3 with a similar analysis. Note that

$$E_n K_n = e^n (m^+)_n^n c^{-1} = \lambda R_{ab}^{nn} (m^+)_n^a e^b c^{-1} = \lambda R_{nn}^{nn} (m^+)_n^n e^n c^{-1} = \lambda \frac{1}{\lambda} R_{nn}^{nn} (m^+)_n^n c^{-1} e^n = q K_n E_n.$$

Combining with the equalities $e^{n-1} = e^n E_{n-1} - q^{-1} E_{n-1} e^n$ and $e^{n-1} E_{n-1} = q E_{n-1} e^{n-1}$ given in [HH1], then we get

$$(E_{n-1})^2 E_n - (q + q^{-1}) E_{n-1} E_n E_{n-1} + E_n (E_{n-1})^2 = 0.$$

On the other hand, combining with the equality $(e^n)^2 e^{n-1} + q e^{n-1} (e^n)^2 = (1+q) e^n e^{n-1} e^n$, we get $(e^n)^2 (e^n E_{n-1} - q^{-1} E_{n-1} e^n) + q (e^n E_{n-1} - q^{-1} E_{n-1} e^n) (e^n)^2 = (1+q) e^n (e^n E_{n-1} - q^{-1} E_{n-1} e^n) e^n$, which can be simplified into $(e^n)^3 E_{n-1} - (q+1+q^{-1}) (e^n)^2 E_{n-1} e^n + (q+1+q^{-1}) e^n E_{n-1} (e^n)^2 - E_{n-1} (e^n)^3 = 0$, namely,

$$(E_n)^3 E_{n-1} - \left[\begin{matrix} 3 \\ 1 \end{matrix} \right]_{q^{\frac{1}{2}}} (E_n)^2 E_{n-1} E_n + \left[\begin{matrix} 3 \\ 2 \end{matrix} \right]_{q^{\frac{1}{2}}} E_n E_{n-1} (E_n)^2 - E_{n-1} (E_n)^3 = 0.$$

The relations between F_{n-1} and F_n can be obtained in the same way. In view of these relations in the new quantum group, the length of the new simple root α_n corresponding to the additional simple root vectors E_n, F_n is $(\alpha_n, \alpha_n) = 1$, and $(\alpha_{n-1}, \alpha_n) = -1$, $(\alpha_j, \alpha_n) = 0$, for $1 \leq j \leq n-2$. This gives rise to the required Cartan matrix of type B_n we want to have.

The proof is complete. \square

3.2. $A_{n-1} \implies C_n$. Let $V = \bigoplus_{i=1}^n kx_i$ denote the representation space of the vector representation T_V of $U_q(\mathfrak{sl}_n)$, which is given by

$$E_i(x_j) = \begin{cases} x_{i+1}, & \text{if } j = i, \\ 0, & \text{otherwise,} \end{cases} \quad F_i(x_j) = \begin{cases} x_i, & \text{if } j = i+1, \\ 0, & \text{otherwise,} \end{cases} \quad K_i(x_j) = \begin{cases} q^{-1}x_i, & \text{if } j = i, \\ qx_{i+1}, & \text{if } j = i+1, \\ x_j, & \text{otherwise.} \end{cases}$$

The corresponding weights of x_1 and x_i are $-\lambda_1, -\lambda_1 + \alpha_1 + \cdots + \alpha_{i-1}, i = 2, \dots, n$. We will consider the quantum 'symmetric square' of the vector representation, denoted by $\text{sym}^2 V$. $\text{sym}^2 V = k\{x_m \otimes x_m, m = 1, \dots, n; x_i \otimes x_j + q^{-1} x_j \otimes x_i, i < j, i, j = 1, \dots, n\}$ is an irreducible $\frac{n(n+1)}{2}$ -dimensional representation of $U_q(\mathfrak{sl}_n)$, given by

$$\begin{aligned} E_k(x_m \otimes x_m) &= \begin{cases} x_m \otimes x_{m+1} + q^{-1} x_{m+1} \otimes x_m, & \text{if } k = m, \\ 0, & \text{otherwise,} \end{cases} \\ F_k(x_m \otimes x_m) &= \begin{cases} x_{m-1} \otimes x_m + q^{-1} x_m \otimes x_{m-1}, & \text{if } k = m-1, \\ 0, & \text{otherwise,} \end{cases} \\ E_k(x_i \otimes x_j + q^{-1} x_j \otimes x_i) &= \begin{cases} (q + q^{-1})x_{i+1} \otimes x_{i+1}, & \text{if } k = i, j = i+1, \\ x_{i+1} \otimes x_j + q^{-1} x_j \otimes x_{i+1}, & \text{if } k = i, j > i+1, \\ x_i \otimes x_{j+1} + q^{-1} x_{j+1} \otimes x_i, & \text{if } k = j, \\ 0, & \text{otherwise,} \end{cases} \\ F_k(x_i \otimes x_j + q^{-1} x_j \otimes x_i) &= \begin{cases} x_{i-1} \otimes x_j + q^{-1} x_j \otimes x_{i-1}, & \text{if } k = i-1, \\ (q + q^{-1})x_i \otimes x_i, & \text{if } k = j-1, j = i+1, \\ x_i \otimes x_{j-1} + q^{-1} x_{j-1} \otimes x_i, & \text{if } k = j-1, j > i+1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

For convenience, denote by $\{v_i\}$ a basis of $\text{sym}^2 V$ with weights $\{\mu_i\}$ arranged in the raising-weights order. With this module $\text{sym}^2 V$, we obtain the corresponding upper triangular R -matrix with respect to the lexicographic order on the basis of $(\text{sym}^2 V)^{\otimes 2}$ induced by the chosen basis of $\text{sym}^2 V$, denoted still by R_{VV} . Note that $E_i^2(x_i \otimes x_i) = (q + q^{-1})x_{i+1} \otimes x_{i+1}$, $i = 1, \dots, n-1$ in $\text{sym}^2 V$, so every E_i^2 is not a zero action. Moreover, the corresponding matrix PR_{VV} is non-symmetric. If $(q - q^{-1})(E_k \otimes F_k)(v_i \otimes v_j) = f(q) \cdot (v_m \otimes v_n)$, for $0 \neq f(q) \in k[q, q^{-1}]$, then $\mu_m = \mu_i + \alpha_k$, $\mu_n = \mu_j - \alpha_k$. Moreover, we have

$$E_k(v_i) = \begin{cases} (q + q^{-1})v_m, & \text{if } v_i = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ v_m, & \text{otherwise,} \end{cases}$$

$$\text{which is equivalent to } F_k(v_m) = \begin{cases} (q + q^{-1})v_i, & \text{if } v_m = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ v_i, & \text{otherwise.} \end{cases}$$

And

$$F_k(v_j) = \begin{cases} (q + q^{-1})v_n, & \text{if } v_j = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ v_n, & \text{otherwise,} \end{cases}$$

$$\text{which is equivalent to } E_k(v_n) = \begin{cases} (q + q^{-1})v_j, & \text{if } v_m = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ v_j, & \text{otherwise.} \end{cases}$$

In view of these facts, we obtain

$$\begin{aligned} & (q - q^{-1})(E_k \otimes F_k)(v_i \otimes v_j) \\ = & \begin{cases} av_m \otimes v_n, & \text{if } v_i = v_j = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ bv_m \otimes v_n & \text{if } v_i = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, v_j \neq x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ & \text{or } v_i \neq x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, v_j = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ (q - q^{-1})v_m \otimes v_n, & \text{otherwise,} \end{cases} \\ & (q - q^{-1})(E_k \otimes F_k)(v_n \otimes v_m) \\ = & \begin{cases} av_j \otimes v_i, & \text{if } v_n = v_m = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ bv_j \otimes v_i & \text{if } v_n = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, v_m \neq x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ & \text{or } v_n \neq x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, v_m = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ (q - q^{-1})v_j \otimes v_i, & \text{otherwise,} \end{cases} \end{aligned}$$

where $a = (q - q^{-1})(q + q^{-1})^2$, $b = (q - q^{-1})(q + q^{-1})$. So we obtain the corresponding entries in the matrix are

$$R_{VV}^{ij} = \begin{cases} aq^{(\mu_m, \mu_n)}, & \text{if } v_i = v_j = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ bq^{(\mu_m, \mu_n)}, & \text{if } v_i = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, v_j \neq x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ & \text{or } v_i \neq x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, v_j = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ (q - q^{-1})q^{(\mu_m, \mu_n)}, & \text{otherwise,} \end{cases}$$

$$R_{VV}^{nm} = \begin{cases} aq^{(\mu_i, \mu_j)}, & \text{if } v_n = v_m = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ bq^{(\mu_i, \mu_j)}, & \text{if } v_n = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, v_m \neq x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ & \text{or } v_n \neq x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, v_m = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k, \\ (q-q^{-1})q^{(\mu_i, \mu_j)}, & \text{otherwise.} \end{cases}$$

With these, we find that R_{VV}^{ij} maybe not equal to R_{VV}^{nm} . For example, when $v_i = v_j = x_k \otimes x_{k+1} + q^{-1}x_{k+1} \otimes x_k$, then $v_m = x_{k+1} \otimes x_{k+1}$, $v_n = x_k \otimes x_k$. we obtain

$$(\mu_m, \mu_n) = (-2\lambda_1 + 2\alpha_1 + \cdots 2\alpha_k, -2\lambda_1 + \cdots 2\alpha_{k-1}) = 4(\lambda_1, \lambda_1) - 4,$$

$$(\mu_i, \mu_j) = (-2\lambda_1 + 2\alpha_1 + \cdots + 2\alpha_{k-1} + \alpha_k, -2\lambda_1 + 2\alpha_1 + \cdots + 2\alpha_{k-1} + \alpha_k) = 4(\lambda_1, \lambda_1) - 2.$$

Then the corresponding entries in the matrix R_{VV} are $R_{VV}^{ij} = (q - q^{-1})(q + q^{-1})^2 q^{(\mu_m, \mu_n)}$, $R_{VV}^{nm} = (q - q^{-1})q^{(\mu_i, \mu_j)}$. Obviously, $R_{VV}^{ij} \neq R_{VV}^{nm} \iff (PR_{VV})_{mn}^{ji} \neq (PR_{VV})_{ji}^{mn}$. Namely, PR_{VV} is non-symmetric.

In order to obtain the dually-paired braided groups, we need to find the matrices R, R' satisfying the conditions in Proposition 2.1. First of all, we have

Proposition 3.2. *Corresponding to the representation $\text{sym}^2 V$, the matrix PR_{VV} obeys the minimal polynomial equation $(PR_{VV} - q^{\frac{4(n-1)}{n}}I)(PR_{VV} - q^{-\frac{2(n+2)}{n}}I)(PR_{VV} + q^{-\frac{4}{n}}I) = 0$.*

Proof. For type A_{n-1} , we know that the decomposition of tensor product for this module is $\text{sym}^2 V \otimes \text{sym}^2 V = V_1 \oplus V_2 \oplus V_3$ (see [FH]), where V_i denotes the irreducible representation with highest weight $2\lambda_2, 2\lambda_1 + \lambda_2, 4\lambda_1$, respectively. This means that there are 3 eigenvalues, denoted by y_1, y_2, y_3 . Set $\mathcal{N} = (PR_{VV} - y_1I)(PR_{VV} - y_2I)(PR_{VV} - y_3I)$, and

$$\Delta_1 = y_1 + y_2 + y_3, \quad \Delta_2 = y_1y_2 + y_1y_3 + y_2y_3, \quad \Delta_3 = y_1y_2y_3.$$

We will consider some special rows. In view of the representation $\text{sym}^2 V$, nonzero entries occurred at rows (12) and (21) in matrix $PR_{VV} - y_iI$ are $(PR_{VV} - y_iI)_{12}^{12} = -y_i$, $(PR_{VV} - y_iI)_{21}^{12} = q^{\frac{2(n-2)}{n}}$, $(PR_{VV} - y_iI)_{12}^{21} = q^{\frac{2(n-2)}{n}}$, $(PR_{VV} - y_iI)_{21}^{21} = q^{\frac{2(n-2)}{n}}(q + q^{-1})(q - q^{-1}) - y_i$. Then the nonzero entries at row (12) in matrix \mathcal{N} are

$$\begin{aligned} \mathcal{N}_{12}^{12} &= -y_1y_2y_3 - q^{\frac{4(n-2)}{n}}(y_1 + y_2 + y_3) + q^{\frac{6(n-2)}{n}}(q^2 - q^{-2}), \\ \mathcal{N}_{21}^{12} &= q^{\frac{2(n-2)}{n}}(y_1y_2 + y_1y_3 + y_2y_3) - q^{\frac{4(n-2)}{n}}(q^2 - q^{-2})(y_1 + y_2 + y_3) + q^{\frac{6(n-2)}{n}}(q^2 - q^{-2})^2. \end{aligned}$$

Nonzero entries at row $(\frac{n(n+1)}{2} - 2, \frac{n(n+1)}{2})$ and $(\frac{n(n+1)}{2}, \frac{n(n+1)}{2} - 2)$ in $PR_{VV} - y_iI$ are

$$\begin{aligned} (PR_{VV} - y_iI)_{\frac{n(n+1)}{2}-2, \frac{n(n+1)}{2}}^{\frac{n(n+1)}{2}-2, \frac{n(n+1)}{2}} &= -y_i, \\ (PR_{VV} - y_iI)_{\frac{n(n+1)}{2}, \frac{n(n+1)}{2}-2}^{\frac{n(n+1)}{2}-2, \frac{n(n+1)}{2}} &= q^{-\frac{4}{n}}, \end{aligned}$$

$$\begin{aligned}
(PR_{VV} - y_i I)_{\frac{n(n+1)}{2}-2, \frac{n(n+1)}{2}-2} &= q^{-\frac{4}{n}}, \\
(PR_{VV} - y_i I)_{\frac{n(n+1)}{2}-1, \frac{n(n+1)}{2}-1} &= q^{\frac{2(n-2)}{n}}(q - q^{-1}), \\
(PR_{VV} - y_i I)_{\frac{n(n+1)}{2}, \frac{n(n+1)}{2}-2} &= q^{\frac{n-4}{n}}(q - q^{-1})^2(q + q^{-1}) - y_i.
\end{aligned}$$

Then the entry at row $(\frac{n(n+1)}{2} - 2, \frac{n(n+1)}{2})$ and column $(\frac{n(n+1)}{2} - 2, \frac{n(n+1)}{2})$ in matrix \mathcal{N} is

$$\mathcal{N}_{\frac{n(n+1)}{2}-2, \frac{n(n+1)}{2}} = -y_1 y_2 y_3 - q^{-\frac{8}{n}}(y_1 + y_2 + y_3) + q^{\frac{n-12}{n}}(q - q^{-1})^2(q + q^{-1}).$$

So we obtain the following equations

$$\begin{cases}
-y_1 y_2 y_3 - q^{\frac{4(n-2)}{n}}(y_1 + y_2 + y_3) + q^{\frac{6(n-2)}{n}}(q^2 - q^{-2}) = 0, \\
q^{\frac{2(n-2)}{n}}(y_1 y_2 + y_1 y_3 + y_2 y_3) - q^{\frac{4(n-2)}{n}}(q^2 - q^{-2})(y_1 + y_2 + y_3) + q^{\frac{6(n-2)}{n}}(q^2 - q^{-2})^2 = 0, \\
-y_1 y_2 y_3 - q^{-\frac{8}{n}}(y_1 + y_2 + y_3) + q^{\frac{n-12}{n}}(q - q^{-1})^2(q + q^{-1}) = 0.
\end{cases}$$

Solving it, we obtain that these eigenvalues are $q^{-\frac{2(n+2)}{n}}, q^{\frac{4(n-1)}{n}}, -q^{\frac{-4}{n}}$. So the minimal polynomial equation of PR_{VV} is $(PR_{VV} - q^{\frac{4(n-1)}{n}} I)(PR_{VV} - q^{-\frac{2(n+2)}{n}} I)(PR_{VV} + q^{\frac{-4}{n}} I) = 0$. \square

Now, set $R = q^{\frac{4}{n}} R_{VV}$, $R' = RPR - (q^{-2} + q^4)R + (q^2 + 1)P$. By Proposition 3.2, we have $(PR + I)(PR' - I) = 0$. Braided (co)vector algebras $V^\vee(R', R_{21}^{-1})$, $V(R', R)$ are braided groups in the braided category $\widetilde{H_{\lambda R}}\mathfrak{M}$, $\mathfrak{M}^{\widetilde{H_{\lambda R}}}$, respectively, and the weakly quasitriangular dual pair $(U_q^{\text{ext}}(\mathfrak{sl}_n), \widetilde{H_{\lambda R}})$ is given by $\langle (m^+)^i_j, t_l^k \rangle = \lambda R_{jl}^{ik}$, $\langle (m^-)^i_j, t_l^k \rangle = \lambda^{-1}(R^{-1})_{lj}^{ki}$, $\langle c, g \rangle = \lambda$, where $\lambda = q^{-\frac{4}{n}}$. The entries in the matrices m^\pm we need can be obtained by Lemma 2.1, listed in the following

Proposition 3.3. *Corresponding to the representation $\text{sym}^2 V$, the diagonal and minor diagonal entries in FRT-matrix m^\pm we need are*

$$\begin{aligned}
(m^+)_2^1 &= (q + q^{-1})(q - q^{-1})E_1(m^+)_2^2, \quad (m^+)_j^{j-1} = (q - q^{-1})E_{j-1}(m^+)_j^j, \quad 3 \leq j \leq n, \\
(m^+)_i^i &= K_1^{\frac{n-2}{n}} \cdots K_{i-1}^{\frac{n-2(i-1)}{n}} K_i^{\frac{2(n-i)}{n}} \cdots K_{n-2}^{\frac{2 \cdot 2}{n}} K_{n-1}^{\frac{2 \cdot 1}{n}}, \quad (m^+)_{\frac{n(n+1)}{2}}^{\frac{n(n+1)}{2}} = K_1^{-\frac{2 \cdot 1}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-1}^{-\frac{2 \cdot (n-1)}{n}}, \\
(m^-)_i^i (m^+)_i^i &= 1, \quad (m^-)_j^{j+1} = q(q - q^{-1})(m^-)_{j+1}^{j+1} F_j, \quad 1 \leq i \leq n, 1 \leq j \leq n-1.
\end{aligned}$$

With these, we have the following

Theorem 3.2. *With $\lambda = q^{-\frac{4}{n}}$, identify $e^{\frac{n(n+1)}{2}}, f_{\frac{n(n+1)}{2}}, (m^+)_{\frac{n(n+1)}{2}} c^{-1}$ with the additional simple root vectors E_n, F_n and the group-like element K_n . Then the resulting new quantum group $U(V^\vee(R_{21}^{-1}, R'), U_q^{\text{ext}}(\mathfrak{sl}_n), V(R, R'))$ is exactly the $U_q(\mathfrak{sp}_{2n})$ with $K_i^{\pm \frac{1}{n}}$ adjoined.*

Proof. For the identification in Theorem 3.2, $[E_n, F_n] = \frac{K_n - K_n^{-1}}{q^2 - q^{-2}}$, $\Delta(E_n) = E_n \otimes K_n + 1 \otimes E_n$, and $\Delta(F_n) = F_n \otimes 1 + K_n^{-1} \otimes F_n$ can be deduced easily from Theorem 2.3. Since the relations of negative part can be obtained in a similar way, we only focus on the relations of the positive part.

$$\begin{aligned} E_n K_n &= e^{\frac{n(n+1)}{2}} (m^+)^{\frac{n(n+1)}{2}} c^{-1} = \lambda R^{\frac{n(n+1)}{2}, \frac{n(n+1)}{2}} (m^+)^{\frac{n(n+1)}{2}} e^{\frac{n(n+1)}{2}} c^{-1} \\ &= \lambda \frac{1}{\lambda} R^{\frac{n(n+1)}{2}, \frac{n(n+1)}{2}} (m^+)^{\frac{n(n+1)}{2}} c^{-1} e^{\frac{n(n+1)}{2}} = q^{\frac{4}{n}} R_{VV}^{\frac{n(n+1)}{2}, \frac{n(n+1)}{2}} K_n E_n \\ &= q^{\frac{4}{n}} q^{\frac{4(n-1)}{n}} K_n E_n = q^4 K_n E_n. \end{aligned}$$

The new group-like element $K_n = (m^+)^{\frac{n(n+1)}{2}} c^{-1} = K_1^{-\frac{2 \cdot 1}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-1}^{-\frac{2 \cdot (n-1)}{n}} c^{-1}$, then when $2 \leq j \leq n-2$, we obtain

$$\begin{aligned} K_n E_j &= K_1^{-\frac{2 \cdot 1}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-1}^{-\frac{2 \cdot (n-1)}{n}} c^{-1} E_j \\ &= q^{-\frac{2(j-1)}{n}} q^{\frac{2j}{n}} q^{-\frac{2(j+1)}{n}} E_j K_1^{-\frac{2 \cdot 1}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-1}^{-\frac{2 \cdot (n-1)}{n}} c^{-1} = E_j K_n, \\ K_n E_1 &= K_1^{-\frac{2 \cdot 1}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-1}^{-\frac{2 \cdot (n-1)}{n}} c^{-1} E_1 \\ &= q^{2 \cdot \frac{2}{n}} q^{-\frac{2 \cdot 2}{n}} E_1 K_1^{-\frac{2 \cdot 1}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-1}^{-\frac{2 \cdot (n-1)}{n}} c^{-1} = E_1 K_n, \\ K_n E_{n-1} &= K_1^{-\frac{2 \cdot 1}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-1}^{-\frac{2 \cdot (n-1)}{n}} c^{-1} E_{n-1} \\ &= q^{-\frac{2 \cdot (n-2)}{n}} q^{2 \cdot \frac{2 \cdot (n-1)}{n}} E_{n-1} K_1^{-\frac{2 \cdot 1}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-1}^{-\frac{2 \cdot (n-1)}{n}} c^{-1} = q^2 E_{n-1} K_n. \end{aligned}$$

In order to explore the relations between the new simple root vector E_n and K_i , $1 \leq i \leq n-1$, we consider the cross relation

$$e^{\frac{n(n+1)}{2}} (m^+)_j^j = \lambda R_{j, \frac{n(n+1)}{2}}^j (m^+)_j^j e^{\frac{n(n+1)}{2}} = R_{VV}^j (m^+)_j^j e^{\frac{n(n+1)}{2}} = q^{(\mu_j, \mu_{\frac{n(n+1)}{2}})} (m^+)_j^j e^{\frac{n(n+1)}{2}}.$$

Combining with $K_i (m^+)_{i+1}^{i+1} = (m^+)_i^i$, $1 \leq i \leq n-1$, which can be deduced from the equalities in the above Proposition, we obtain $e^{\frac{n(n+1)}{2}} K_i = q^{(\mu_i - \mu_{i+1}, \mu_{\frac{n(n+1)}{2}})} K_i e^{\frac{n(n+1)}{2}}$, which is equivalent to $E_n K_i = q^{(\mu_i - \mu_{i+1}, \mu_{\frac{n(n+1)}{2}})} K_i E_n$. According to the weight of x_i , we get $\mu_i = -2\lambda_1 + \alpha_1 + \cdots + \alpha_{i-1}$, $\mu_{i+1} = -2\lambda_1 + \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i$, $\mu_{\frac{n(n+1)}{2}} = -2\lambda_1 + 2\alpha_1 + \cdots + 2\alpha_{n-2} + 2\alpha_{n-1}$, then

$$q^{(\mu_i - \mu_{i+1}, \mu_{\frac{n(n+1)}{2}})} = q^{(-\alpha_i, -2\lambda_1 + 2\alpha_1 + \cdots + 2\alpha_{n-2} + 2\alpha_{n-1})} = \begin{cases} 1, & 1 \leq i \leq n-2, \\ q^{-2}, & i = n-1. \end{cases}$$

So $K_i E_n = E_n K_i$, $1 \leq i \leq n-2$; $E_n K_{n-1} = q^2 K_{n-1} E_n$.

We also observe that each E_i is included in the entry $(m^+)_{i+1}^i$ by the above Proposition, then the q -Serre relations between $e^{\frac{n(n+1)}{2}}$ and E_i , $1 \leq i \leq n-1$ can be deduced from the

following equalities

$$\begin{cases} e^{\frac{n(n+1)}{2}}(m^+)_{i+1}^i = \lambda R_{i, \frac{n(n+1)}{2}}^{i, \frac{n(n+1)}{2}}(m^+)_{i+1}^i e^{\frac{n(n+1)}{2}} + \lambda R_{i+1, \frac{n(n+1)}{2}-1}^{i, \frac{n(n+1)}{2}}(m^+)_{i+1}^{i+1} e^{\frac{n(n+1)}{2}-1}, \\ e^{\frac{n(n+1)}{2}}(m^+)_{i+1}^{i+1} = \lambda R_{i+1, \frac{n(n+1)}{2}}^{i+1, \frac{n(n+1)}{2}}(m^+)_{i+1}^{i+1} e^{\frac{n(n+1)}{2}}, \\ (m^+)_2^1 = (q + q^{-1})(q - q^{-1})E_1(m^+)_2^2, \\ (m^+)_{i+1}^i = (q - q^{-1})E_i(m^+)_{i+1}^{i+1}, \quad 2 \leq i \leq n-1, \\ R_{i+1, \frac{n(n+1)}{2}-1}^{i, \frac{n(n+1)}{2}} = 0, \quad 1 \leq i \leq n-2. \end{cases}$$

In virtue of these, we obtain

$$\begin{cases} e^{\frac{n(n+1)}{2}}E_i = E_i e^{\frac{n(n+1)}{2}}, \\ e^{\frac{n(n+1)}{2}-1} = e^{\frac{n(n+1)}{2}}E_{n-1} - q^{-2}E_{n-1}e^{\frac{n(n+1)}{2}}, \end{cases} \iff \begin{cases} E_n E_i = E_i E_n, \quad i < n, \\ e^{\frac{n(n+1)}{2}-1} = E_n E_{n-1} - q^{-2}E_{n-1}E_n. \end{cases}$$

So we need to explore the relations between $e^{\frac{n(n+1)}{2}-1}$ and $e^{\frac{n(n+1)}{2}}, E_{n-1}$. According to $R' = RPR - (q^{-2} + q^4)R + (q^2 + 1)P$, we get $R'_{\frac{n(n+1)}{2}, \frac{n(n+1)}{2}-1} = -q^2 - 1$ and $R'_{\frac{n(n+1)}{2}-1, \frac{n(n+1)}{2}} = q^4 + q^2 + 1$. Then

$$e^{\frac{n(n+1)}{2}-1}e^{\frac{n(n+1)}{2}} = R'_{a,b} e^a e^b = -(q^2 + 1)e^{\frac{n(n+1)}{2}}e^{\frac{n(n+1)}{2}-1} + (q^4 + q^2 + 1)e^{\frac{n(n+1)}{2}-1}e^{\frac{n(n+1)}{2}},$$

so $e^{\frac{n(n+1)}{2}}e^{\frac{n(n+1)}{2}-1} = q^2 e^{\frac{n(n+1)}{2}-1}e^{\frac{n(n+1)}{2}}$. Combining with $e^{\frac{n(n+1)}{2}-1} = E_n E_{n-1} - q^{-2}E_{n-1}E_n$, we obtain

$$(E_n)^2 E_{n-1} - (q^2 + q^{-2})E_n E_{n-1} E_n + E_{n-1}(E_n)^2 = 0.$$

On the other hand, according to

$$e^{\frac{n(n+1)}{2}-1}(m^+)_n^{n-1} = \lambda R_{n-1, \frac{n(n+1)}{2}-1}^{n-1, \frac{n(n+1)}{2}-1}(m^+)_n^{n-1} e^{\frac{n(n+1)}{2}-1} + \lambda R_{n, \frac{n(n+1)}{2}-2}^{n-1, \frac{n(n+1)}{2}-1}(m^+)_n^n e^{\frac{n(n+1)}{2}-2},$$

we obtain $e^{\frac{n(n+1)}{2}-2} = \frac{1}{q+q^{-1}}(e^{\frac{n(n+1)}{2}-1}E_{n-1} - E_{n-1}e^{\frac{n(n+1)}{2}-1})$. So the relation between $e^{\frac{n(n+1)}{2}-1}$

and E_{n-1} depends on the relation between $e^{\frac{n(n+1)}{2}-2}$ and E_{n-1} , and the relation between $e^{\frac{n(n+1)}{2}-2}$ and E_{n-1} is $e^{\frac{n(n+1)}{2}-2}E_{n-1} = q^2 E_{n-1}e^{\frac{n(n+1)}{2}-2}$, which is deduced from the equality

$$e^{\frac{n(n+1)}{2}-2}(m^+)_n^{n-1} = \lambda R_{n-1, \frac{n(n+1)}{2}-2}^{n-1, \frac{n(n+1)}{2}-2}(m^+)_n^{n-1} e^{\frac{n(n+1)}{2}-2}. \text{ Then we obtain}$$

$$(e^{\frac{n(n+1)}{2}-1}E_{n-1} - E_{n-1}e^{\frac{n(n+1)}{2}-1})E_{n-1} = q^2 E_{n-1}(e^{\frac{n(n+1)}{2}-1}E_{n-1} - E_{n-1}e^{\frac{n(n+1)}{2}-1}).$$

Combining with $e^{\frac{n(n+1)}{2}-1} = E_n E_{n-1} - q^{-2}E_{n-1}E_n$ again, we obtain

$$(E_{n-1})^3 E_n - (q^2 + 1 + q^{-2})(E_{n-1})^2 E_n E_{n-1} + (q^2 + 1 + q^{-2})E_{n-1}E_n E_{n-1}^2 - E_n(E_{n-1})^3 = 0,$$

namely,

$$(E_{n-1})^3 E_n - \left[\begin{matrix} 3 \\ 1 \end{matrix} \right]_q (E_{n-1})^2 E_n E_{n-1} + \left[\begin{matrix} 3 \\ 2 \end{matrix} \right]_q E_{n-1}E_n E_{n-1}^2 - E_n(E_{n-1})^3 = 0.$$

In view of these relations, the length of the new simple root α_n corresponding to the additional simple root vectors E_n, F_n is $(\alpha_n, \alpha_n) = 4$, $(\alpha_{n-1}, \alpha_n) = -2$, and $(\alpha_i, \alpha_n) = 0$, $1 \leq i \leq n-2$. This gives us the Cartan matrix of type C_n . The proof is complete. \square

3.3. $A_{n-1} \implies D_n$. Let $\wedge^2 V$ denote the second quantum exterior power of the vector representation T_V of $U_q(\mathfrak{sl}_n)$, then $\wedge^2 V \cong (V \otimes V)/\text{sym}^2 V = \bigoplus_{i,j=1}^n k\{x_i \wedge x_j, i < j\}$, and $x_i \wedge x_i = 0, x_i \wedge x_j = -q^{-1}x_j \wedge x_i, i < j$. $\wedge^2 V$ is an irreducible $\frac{n(n-1)}{2}$ -dimensional representation of $U_q(\mathfrak{sl}_n)$, given by

$$E_k(x_i \wedge x_j) = \begin{cases} x_{i+1} \wedge x_j, & \text{if } k = i, j > i + 1, \\ x_i \wedge x_{j+1}, & \text{if } k = j, \\ 0, & \text{otherwise,} \end{cases}$$

$$F_k(x_i \wedge x_j) = \begin{cases} x_{i-1} \wedge x_j, & \text{if } k = i - 1, \\ x_i \wedge x_{j-1}, & \text{if } k = j - 1, j > i + 1, \\ 0, & \text{otherwise.} \end{cases}$$

For convenience, let $\{v_i\}$ be a basis of $\wedge^2 V$ with weights $\{\mu_i\}$ arranged in the raising-weights order. Associated to this $\frac{n(n-1)}{2}$ -dimensional module $\wedge^2 V$, we see that $E_\beta(f_m) = f_n \iff F_\beta(f_n) = f_m$ and every E_i^2 is zero action. With these, we obtain the following

Proposition 3.4. (1) *The matrix PR_{VV} is symmetric and obeys the minimal polynomial equation*

$$(PR_{VV} - q^{\frac{2(n-2)}{n}} I)(PR_{VV} - q^{-\frac{4}{n}} I)(PR_{VV} + q^{-\frac{4}{n}} I) = 0.$$

(2) *The diagonal and minor diagonal entries in FRT-matrix m^\pm we need are*

$$(m^+)_i^{i-1} = (q - q^{-1})E_i(m^+)_i^i, \quad (m^-)_{i-1}^i = q(q - q^{-1})(m^-)_i^i F_i, \quad 2 \leq i \leq n-1.$$

$$(m^+)_i^i = K_1^{\frac{n-2}{n}} \cdots K_i^{\frac{n-2i}{n}} K_{i+1}^{\frac{2(n-(i+1))}{n}} \cdots K_{n-2}^{\frac{2 \cdot 2}{n}} K_{n-1}^{\frac{2 \cdot 1}{n}}, \quad (m^-)_i^i (m^+)_i^i = 1, \quad 1 \leq i \leq n-1.$$

$$(m^+)_{2n-3}^{n-1} = (q - q^{-1})E_1 K_1^{-1} (m^+)_{n-1}^{n-1} = (q - q^{-1})E_1 K_1^{-\frac{2}{n}} K_2^{\frac{n-2 \cdot 2}{n}} K_3^{\frac{n-2 \cdot 3}{n}} \cdots K_{n-2}^{\frac{n-2 \cdot (n-2)}{n}} K_{n-1}^{\frac{n-2 \cdot (n-1)}{n}}.$$

$$(m^-)_{n-1}^{2n-3} = q(q - q^{-1})(m^-)_{n-2}^{n-2} K_1 F_1 = q(q - q^{-1})K_1^{\frac{2}{n}} K_2^{-\frac{n-2 \cdot 2}{n}} K_3^{-\frac{n-2 \cdot 3}{n}} \cdots K_{n-2}^{-\frac{n-2 \cdot (n-2)}{n}} K_{n-1}^{-\frac{n-2 \cdot (n-1)}{n}} F_1.$$

$$(m^+)_{\frac{n(n-1)}{2}}^{\frac{n(n-1)}{2}} = K_1^{-\frac{2}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-3}^{-\frac{2 \cdot (n-3)}{n}} K_{n-2}^{-\frac{2 \cdot (n-2)}{n}} K_{n-1}^{-\frac{n-2}{n}}.$$

Proof. These results can be obtained by the similar methods. We only describe the minimal polynomial in (1), which is deduced by the ingenious method in Proposition 3.2. Firstly, it is well-known that for type A_{n-1} , when $n \geq 4$, the decomposition of the tensor product of the module is $\wedge^2 V \otimes \wedge^2 V = V_1 \oplus V_2 \oplus V_3$ (see [FH]), where V_i is the irreducible representation with highest weight $2\lambda_2, \lambda_1 + \lambda_3, \lambda_4$, respectively. This means that there are 3 eigenvalues,

denoted by y_1, y_2, y_3 . We will also consider nonzero entries at rows (12), (21) in matrix $PR_{VV} - y_i I$, which are $(PR_{VV} - y_i I)_{12}^{12} = -y_i$, $(PR_{VV} - y_i I)_{21}^{12} = q^{\frac{n-4}{n}}$, $(PR_{VV} - y_i I)_{12}^{21} = q^{\frac{n-4}{n}}$, $(PR_{VV} - y_i I)_{21}^{21} = q^{\frac{n-4}{n}}(q - q^{-1}) - y_i$. Then the entries at row (12) in matrix \mathcal{N} are

$$\begin{aligned}\mathcal{N}_{12}^{12} &= -y_1 y_2 y_3 - q^{\frac{2(n-4)}{n}}(y_1 + y_2 + y_3) + q^{\frac{2(n-4)}{n}}(q - q^{-1}), \\ \mathcal{N}_{21}^{12} &= q^{\frac{n-4}{n}}(y_1 y_2 + y_2 y_3 + y_1 y_3) - q^{\frac{2(n-4)}{n}}(q - q^{-1})(y_1 + y_2 + y_3) + q^{\frac{3(n-4)}{n}}[1 + (q - q^{-1})^2].\end{aligned}$$

Nonzero entries at row $(1, \frac{n(n-1)}{2})$ in matrix $PR_{VV} - y_i I$ are $(PR_{VV} - y_i I)_{1, \frac{n(n-1)}{2}}^{1, \frac{n(n-1)}{2}} = -y_i$, $(PR_{VV} - y_i I)_{\frac{n(n-1)}{2}, 1}^{1, \frac{n(n-1)}{2}} = q^{-\frac{4}{n}}$. Nonzero entries at row $(\frac{n(n-1)}{2}, 1)$ in matrix $PR_{VV} - y_i I$ are $(PR_{VV} - y_i I)_{1, \frac{n(n-1)}{2}}^{\frac{n(n-1)}{2}, 1} = q^{-\frac{4}{n}}$, $(PR_{VV} - y_i I)_{\frac{n(n-1)}{2}, 1}^{\frac{n(n-1)}{2}, 1} = -y_i$, moreover, there exist some a_j, b_j , $1 < a_j, b_j < \frac{n(n-1)}{2}$, and $(PR_{VV} - y_i I)_{a_j, b_j}^{\frac{n(n-1)}{2}, 1} = f(q)$, for some $f(q) \in k[q, q^{-1}]$. Then we obtain the entries at row $(1, \frac{n(n-1)}{2})$ in matrix $(PR_{VV} - y_1 I)(PR_{VV} - y_2 I)$ are

$$\begin{aligned}& [(PR_{VV} - y_1 I)(PR_{VV} - y_2 I)]_{1, \frac{n(n-1)}{2}}^{1, \frac{n(n-1)}{2}} \\ &= (PR_{VV} - y_1 I)_{1, \frac{n(n-1)}{2}}^{1, \frac{n(n-1)}{2}} (PR_{VV} - y_2 I)_{1, \frac{n(n-1)}{2}}^{1, \frac{n(n-1)}{2}} + (PR_{VV} - y_1 I)_{\frac{n(n-1)}{2}, 1}^{1, \frac{n(n-1)}{2}} (PR_{VV} - y_2 I)_{1, \frac{n(n-1)}{2}}^{\frac{n(n-1)}{2}, 1} \\ &= y_1 y_2 + q^{-\frac{8}{n}},\end{aligned}$$

$$\begin{aligned}& [(PR_{VV} - y_1 I)(PR_{VV} - y_2 I)]_{\frac{n(n-1)}{2}, 1}^{1, \frac{n(n-1)}{2}} \\ &= (PR_{VV} - y_1 I)_{1, \frac{n(n-1)}{2}}^{1, \frac{n(n-1)}{2}} (PR_{VV} - y_2 I)_{\frac{n(n-1)}{2}, 1}^{1, \frac{n(n-1)}{2}} + (PR_{VV} - y_1 I)_{\frac{n(n-1)}{2}, 1}^{1, \frac{n(n-1)}{2}} (PR_{VV} - y_2 I)_{\frac{n(n-1)}{2}, 1}^{\frac{n(n-1)}{2}, 1} \\ &= -q^{-\frac{4}{n}}(y_1 + y_2),\end{aligned}$$

$$\begin{aligned}& [(PR_{VV} - y_1 I)(PR_{VV} - y_2 I)]_{a_j, b_j}^{1, \frac{n(n-1)}{2}} \\ &= (PR_{VV} - y_1 I)_{1, \frac{n(n-1)}{2}}^{1, \frac{n(n-1)}{2}} (PR_{VV} - y_2 I)_{a_j, b_j}^{1, \frac{n(n-1)}{2}} + (PR_{VV} - y_1 I)_{\frac{n(n-1)}{2}, 1}^{1, \frac{n(n-1)}{2}} (PR_{VV} - y_2 I)_{a_j, b_j}^{\frac{n(n-1)}{2}, 1} \\ &= (PR_{VV} - y_1 I)_{\frac{n(n-1)}{2}, 1}^{1, \frac{n(n-1)}{2}} (PR_{VV} - y_2 I)_{a_j, b_j}^{\frac{n(n-1)}{2}, 1} = q^{-\frac{4}{n}} f(q).\end{aligned}$$

With these entries, we obtain the entry at row $(1, \frac{n(n-1)}{2})$ and column $(1, \frac{n(n-1)}{2})$ in matrix \mathcal{N} is

$$\begin{aligned}\mathcal{N}_{1, \frac{n(n-1)}{2}}^{1, \frac{n(n-1)}{2}} &= [(PR_{VV} - y_1 I)(PR_{VV} - y_2 I)]_{1, \frac{n(n-1)}{2}}^{1, \frac{n(n-1)}{2}} (PR_{VV} - y_3 I)_{1, \frac{n(n-1)}{2}}^{1, \frac{n(n-1)}{2}} \\ &\quad + [(PR_{VV} - y_1 I)(PR_{VV} - y_2 I)]_{\frac{n(n-1)}{2}, 1}^{1, \frac{n(n-1)}{2}} (PR_{VV} - y_3 I)_{1, \frac{n(n-1)}{2}}^{\frac{n(n-1)}{2}, 1} \\ &\quad + [(PR_{VV} - y_1 I)(PR_{VV} - y_2 I)]_{a_j, b_j}^{1, \frac{n(n-1)}{2}} (PR_{VV} - y_3 I)_{1, \frac{n(n-1)}{2}}^{a_j, b_j} \\ &= -(y_1 y_2 + q^{-\frac{8}{n}})y_3 - q^{-\frac{4}{n}}(y_1 + y_2)q^{-\frac{4}{n}} + q^{-\frac{4}{n}} f(q) \cdot 0 \\ &= -y_1 y_2 y_3 - q^{-\frac{8}{n}}(y_1 + y_2 + y_3).\end{aligned}$$

With these, we obtain the following equations

$$\begin{cases} -y_1 y_2 y_3 - q^{\frac{2(n-4)}{n}}(y_1 + y_2 + y_3) + q^{\frac{2(n-4)}{n}}(q - q^{-1}) = 0, \\ q^{\frac{n-4}{n}}(y_1 y_2 + y_2 y_3 + y_1 y_3) - q^{\frac{2(n-4)}{n}}(q - q^{-1})(y_1 + y_2 + y_3) + q^{\frac{3(n-4)}{n}}[1 + (q - q^{-1})^2] = 0, \\ -y_1 y_2 y_3 - q^{-\frac{8}{n}}(y_1 + y_2 + y_3) = 0. \end{cases}$$

Solving it, we obtain that these eigenvalues are $q^{\frac{2(n-2)}{n}}, \pm q^{-\frac{4}{n}}$. So the minimal polynomial equation of PR_{VV} is $(PR_{VV} - q^{\frac{2(n-2)}{n}}I)(PR_{VV} - q^{-\frac{4}{n}}I)(PR_{VV} + q^{-\frac{4}{n}}I) = 0$. \square

Set $R = q^{\frac{4}{n}}R_{VV}$, $R' = RPR - (q^2+1)R + (q^2+1)P$, then we have $(PR+I)(PR'-I) = 0$ by Proposition 3.4, and braided groups $V^\vee(R', R_{21}^{-1}), V(R', R)$ in the braided category $\widetilde{H_{AR}}\mathfrak{M}, \mathfrak{M}\widetilde{H_{AR}}$, respectively. With these analysis, we obtain the following

Theorem 3.3. *With $\lambda = q^{-\frac{4}{n}}$, identify $e^{\frac{n(n-1)}{2}}, f_{\frac{n(n-1)}{2}}, (m^+)^{\frac{n(n-1)}{2}}c^{-1}$ with the additional simple root vectors E_n, F_n and the group-like element K_n . Then the resulting new quantum group $U(V^\vee(R', R_{21}^{-1}), U_q^{ext}(\mathfrak{sl}_n), V(R', R))$ is exactly the $U_q(\mathfrak{so}_{2n})$ with $K_i^{\pm \frac{1}{n}}$ adjoined.*

Proof. For the identification in Theorem 3.3, $[E_n, F_n] = \frac{K_n - K_n^{-1}}{q - q^{-1}}, \Delta(E_n) = E_n \otimes K_n + 1 \otimes E_n$, and $\Delta(F_n) = F_n \otimes 1 + K_n^{-1} \otimes F_n$ are deduced easily from Theorem 2.3.

$$\begin{aligned} E_n K_n &= e^{\frac{n(n-1)}{2}} (m^+)^{\frac{n(n-1)}{2}} c^{-1} = \lambda R^{\frac{n(n-1)}{2}, \frac{n(n-1)}{2}} (m^+)^{\frac{n(n-1)}{2}} e^{\frac{n(n-1)}{2}} c^{-1} \\ &= \lambda \frac{1}{\lambda} R^{\frac{n(n-1)}{2}, \frac{n(n-1)}{2}} (m^+)^{\frac{n(n-1)}{2}} c^{-1} e^{\frac{n(n-1)}{2}} = q^{\frac{4}{n}} R_{VV}^{\frac{n(n-1)}{2}, \frac{n(n-1)}{2}} K_n E_n \\ &= q^{\frac{4}{n}} q^{\frac{2n-4}{n}} K_n E_n = q^2 K_n E_n. \end{aligned}$$

We will consider the relations between K_n and $E_i, 1 \leq i \leq n-1$. According to the equality $(m^+)^{\frac{n(n-1)}{2}} = K_1^{-\frac{2}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-3}^{-\frac{2 \cdot (n-3)}{n}} K_{n-2}^{-\frac{2 \cdot (n-2)}{n}} K_{n-1}^{-\frac{n-2}{n}}$, when $1 \leq j \leq n-3$, we obtain

$$\begin{aligned} K_n E_j &= K_1^{-\frac{2}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-3}^{-\frac{2 \cdot (n-3)}{n}} K_{n-2}^{-\frac{2 \cdot (n-2)}{n}} K_{n-1}^{-\frac{n-2}{n}} c^{-1} E_j \\ &= q^{-\frac{2 \cdot (j-1)}{n}} q^{\frac{2 \cdot j}{n}} q^{-\frac{2 \cdot (j+1)}{n}} E_j K_1^{-\frac{2}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-3}^{-\frac{2 \cdot (n-3)}{n}} K_{n-2}^{-\frac{2 \cdot (n-2)}{n}} K_{n-1}^{-\frac{n-2}{n}} c^{-1} = E_j K_n, \end{aligned}$$

$$\begin{aligned} K_n E_{n-2} &= K_1^{-\frac{2}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-3}^{-\frac{2 \cdot (n-3)}{n}} K_{n-2}^{-\frac{2 \cdot (n-2)}{n}} K_{n-1}^{-\frac{n-2}{n}} c^{-1} E_{n-2} \\ &= q^{-\frac{2 \cdot (n-3)}{n}} q^{\frac{2 \cdot (n-2)}{n}} q^{-\frac{n-2}{n}} E_{n-2} K_1^{-\frac{2}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-3}^{-\frac{2 \cdot (n-3)}{n}} K_{n-2}^{-\frac{2 \cdot (n-2)}{n}} K_{n-1}^{-\frac{n-2}{n}} c^{-1} = q E_{n-2} K_n, \end{aligned}$$

$$\begin{aligned} K_n E_{n-1} &= K_1^{-\frac{2}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-3}^{-\frac{2 \cdot (n-3)}{n}} K_{n-2}^{-\frac{2 \cdot (n-2)}{n}} K_{n-1}^{-\frac{n-2}{n}} c^{-1} E_{n-1} \\ &= q^{-\frac{2 \cdot (n-2)}{n}} q^{\frac{n-2}{n}} E_{n-1} K_1^{-\frac{2}{n}} K_2^{-\frac{2 \cdot 2}{n}} \cdots K_{n-3}^{-\frac{2 \cdot (n-3)}{n}} K_{n-2}^{-\frac{2 \cdot (n-2)}{n}} K_{n-1}^{-\frac{n-2}{n}} c^{-1} = E_{n-1} K_n. \end{aligned}$$

The cross relations between the new simple root vector E_n and K_i , $1 \leq i \leq n-1$ can be obtained by the following equalities

$$\begin{cases} K_i(m^+)_i^i = (m^+)_{i-1}^{i-1}, & 2 \leq i \leq n-1, \\ e^{\frac{n(n-1)}{2}}(m^+)_j^j = \lambda R_{j, \frac{n(n-1)}{2}}^j(m^+)_j^j e^{\frac{n(n-1)}{2}} = q^{(\mu_j, \mu_{\frac{n(n-1)}{2}})}(m^+)_j^j e^{\frac{n(n-1)}{2}}, & \text{for any } j. \end{cases}$$

So $e^{\frac{n(n-1)}{2}} K_i = q^{(\mu_{i-1} - \mu_i, \mu_{\frac{n(n-1)}{2}})} K_i e^{\frac{n(n-1)}{2}}$, and according to

$$q^{(\mu_{i-1} - \mu_i, \mu_{\frac{n(n-1)}{2}})} = q^{(-\alpha_i, -2\lambda_1 + 2\alpha_1 + \dots + 2\alpha_{n-2} + \alpha_{n-1})} = \begin{cases} 1, & 2 \leq i \leq n-3, \\ q^{-1}, & i = n-2, \\ 1, & i = n-1, \end{cases}$$

then we obtain $\begin{cases} E_n K_i = K_i E_n, & 2 \leq i \leq n-3, \\ E_n K_{n-2} = q^{-1} K_{n-2} E_n, \\ E_n K_{n-1} = K_{n-1} E_n. \end{cases}$

Associated with $E_n K_n = q^2 K_n E_n$, we obtain $E_n K_1 = K_1 E_n$. We will explore the q -Serre relations between E_n and E_i , $1 \leq i \leq n-1$. We see that E_1 is included in the $(m^+)_{2n-3}^{n-1}$, so the relation $E_n E_1 = E_1 E_n$ can be deduced from

$$e^{\frac{n(n-1)}{2}}(m^+)_{2n-3}^{n-1} = \lambda R_{n-1, \frac{n(n-1)}{2}}^{n-1}(m^+)_{2n-3}^{n-1} e^{\frac{n(n-1)}{2}} = q^{(\mu_{n-1}, \mu_{\frac{n(n-1)}{2}})}(m^+)_{2n-3}^{n-1} e^{\frac{n(n-1)}{2}}.$$

The other q -Serre relations can be deduced from the following relations

$$\begin{cases} e^{\frac{n(n-1)}{2}}(m^+)_i^{i-1} = \lambda R_{i-1, \frac{n(n-1)}{2}}^{i-1}(m^+)_i^{i-1} e^{\frac{n(n-1)}{2}}, & 2 \leq i \leq n-1, \text{ and } i \neq n-2, \\ e^{\frac{n(n-1)}{2}}(m^+)_{n-2}^{n-3} = \lambda R_{n-3, \frac{n(n-1)}{2}}^{n-3}(m^+)_{n-2}^{n-3} e^{\frac{n(n-1)}{2}} + \lambda R_{n-2, \frac{n(n-1)}{2}}^{n-3}(m^+)_{n-2}^{n-2} e^{\frac{n(n-1)}{2}-1}, \\ (m^+)_i^{i-1} = (q - q^{-1})E_i(m^+)_i^i, & 2 \leq i \leq n-1. \end{cases}$$

Then we obtain $\begin{cases} E_n E_i = E_i E_n, & 2 \leq i \leq n-1, \text{ and } i \neq n-2, \\ e^{\frac{n(n-1)}{2}-1} = E_n E_{n-2} - q^{-1} E_{n-2} E_n. \end{cases}$

In order to explore the relation of $e^{\frac{n(n-1)}{2}}$ and E_{n-2} , we need to know the relations between $e^{\frac{n(n-1)}{2}}$ and $e^{\frac{n(n-1)}{2}-1}$, E_{n-2} . From $R' = RPR - (q^2 + 1)R + (q^2 + 1)P$, we get

$$R'^{\frac{n(n-1)}{2}, \frac{n(n-1)}{2}-1} = 2q^2 + 1, \quad R'^{\frac{n(n-1)}{2}, \frac{n(n-1)}{2}-1} = -2q,$$

then $e^{\frac{n(n-1)}{2}-1} e^{\frac{n(n-1)}{2}} = R'_{a, b}^{\frac{n(n-1)}{2}, \frac{n(n-1)}{2}-1} e^a e^b = (2q^2 + 1)e^{\frac{n(n-1)}{2}-1} e^{\frac{n(n-1)}{2}} - 2qe^{\frac{n(n-1)}{2}} e^{\frac{n(n-1)}{2}-1}$, so $e^{\frac{n(n-1)}{2}} e^{\frac{n(n-1)}{2}-1} = qe^{\frac{n(n-1)}{2}-1} e^{\frac{n(n-1)}{2}}$. Combining with $e^{\frac{n(n-1)}{2}-1} = E_n E_{n-2} - q^{-1} E_{n-2} E_n$, we obtain

$$(E_n)^2 E_{n-2} - (q + q^{-1}) E_n E_{n-2} E_n + E_{n-2} (E_n)^2 = 0.$$

On the other hand, the relation between $e^{\frac{n(n-1)}{2}-1}$ and E_{n-2} is $e^{\frac{n(n-1)}{2}-1}E_{n-2} = qE_{n-2}e^{\frac{n(n-1)}{2}-1}$, which is deduced from $e^{\frac{n(n-1)}{2}-1}(m^+)_{n-2}^{n-3} = \lambda R_{n-3, \frac{n(n-1)}{2}-1}^{n-3}(m^+)_{n-2}^{n-3}e^{\frac{n(n-1)}{2}-1}$. Combining with $e^{\frac{n(n-1)}{2}-1} = E_n E_{n-2} - q^{-1}E_{n-2}E_n$ again, we obtain

$$(E_{n-2})^2 E_n - (q + q^{-1})E_{n-2}E_n E_{n-2} + E_n(E_{n-2})^2 = 0.$$

The relations of negative part can be obtained by the similar analysis.

The length of the new simple root α_n corresponding to the additional simple root vector E_n, F_n is $(\alpha_n, \alpha_n) = 2$, $(\alpha_{n-1}, \alpha_n) = 0$, $(\alpha_{n-2}, \alpha_n) = -1$, and $(\alpha_i, \alpha_n) = 0$, $1 \leq i \leq n-3$. This leads to the Cartan matrix of type D_n .

The proof is complete. \square

4. HOW THE TREE OF QUANTUM GROUPS GROWS UP BY DOUBLE-BOSONIZATION PROCEDURE

As a summary of the results in section 3, together with the results of [HH1, HH2, HH3] we have obtained, we also have a root-system expression for the double-bosonization recursive constructions of $U_q(\mathfrak{g})$'s for the finite-dimensional complex simple Lie algebras \mathfrak{g} .

Proposition 4.1. *Let T_V be an irreducible p -dimensional representation of $U_q(\mathfrak{g})$ with highest weight $-\mu$ (the difference here from [Ro] results from that we use the Majid's version of $U_q(\mathfrak{g})$ whose differences with its standard version lie $K_i^\pm \mapsto K_i^\mp$ and $E_i \mapsto -E_i$), v the weight of the central element c^{-1} , $(a_{ij})_{(n-1) \times (n-1)}$ the Cartan matrix of \mathfrak{g} . Then the corresponding Cartan matrix of the new quantum group $U(V^\vee(R', R_{21}^{-1}), \widehat{U_q^{ext}(\mathfrak{g})}, V(R', R))$ is of a higher rank one, which is obtained from $(a_{ij})_{(n-1) \times (n-1)}$ by adding a row and a column with: $a_{i,n} = \frac{2(\alpha_i, \mu)}{(\alpha_i, \alpha_i)}$, $a_{n,i} = \frac{2(\mu, \alpha_i)}{(\mu, \mu) + (v, v)}$, and v is orthogonal to μ and α_i , $i = 1, \dots, n-1$.*

Proof. From the recursive constructions in [HH1, HH2, HH3] and the results in section 3 of this paper, we know that the new additional group-like element K_n is the element $(m^+)_p^p c^{-1}$. From the expression of $(m^+)_p^p$ in each case, we observe that the corresponding weight of $(m^+)_p^p$ is μ , namely $(m^+)_p^p = K_\mu$, so $K_n = K_{\mu+v}$. Every simple root vector E_j normally locates in the minor diagonal entry $(m^+)_{i+1}^i$, and $(m^+)_{i+1}^i = a_i E_j (m^+)_{i+1}^{i+1}$, $a_i \in k[q, q^{-1}]$. According to the cross relations in Theorem 2.3, we obtain the following relations between e^i and e^{i-1} .

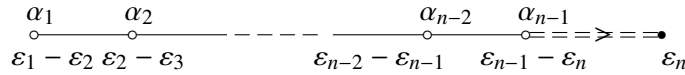
$$\begin{cases} e^i (m^+)_{i+1}^i = \lambda R_{ab}^{ii} (m^+)_{i+1}^a e^b = \lambda R_{ii}^{ii} (m^+)_{i+1}^i e^i + \lambda R_{i+1 i-1}^{ii} (m^+)_{i+1}^{i+1} e^{i-1}, \\ e^i (m^+)_{i+1}^{i+1} = \lambda R_{i+1 i}^{i+1 i} (m^+)_{i+1}^{i+1} e^i, \\ e^{i-1} (m^+)_{i+1}^{i+1} = \lambda R_{i+1 i-1}^{i+1 i-1} (m^+)_{i+1}^{i+1} e^{i-1}. \end{cases}$$

Combining with $(m^+)_{i+1}^i = a_i E_j (m^+)_{i+1}^{i+1}$, we get $e^{i-1} = a_i \frac{R_{i+1 i-1}^{i+1 i-1}}{R_{i+1 i}^{i+1 i}} (e^i E_j - \frac{R_{ii}^{ii}}{R_{i+1 i}^{i+1 i}} E_j e^i)$. Then $\text{wt}(e^{i-1}) = \text{wt}(e^i) + \alpha_j$, combining with $e^k \triangleleft c^{-1} = \lambda^{-1} e^k$ for any k , so $(v, \alpha_j) = 0$. On the

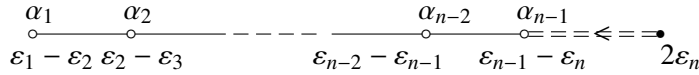
other hand, we know the explicit form of the additional group-like element $K_{\mu+\nu}$ in each recursive construction, so we can obtain the specific form of ν , and ν is orthogonal to μ . \square

In this way, we will give the specific form of weights μ and ν for the above three cases in section 3. The imaginary line and the filled circle mean that the Dynkin diagram A_{n-1} extends to the new added simple root with an arrow pointing to the shorter of the two roots.

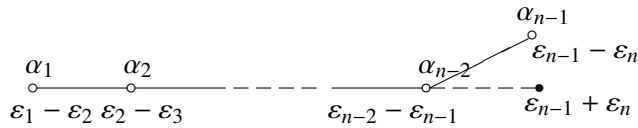
(1) The B_n series. Take $\mathfrak{g} = \mathfrak{sl}_n$ and V the first fundamental representation (i.e., the vector representation), with lowest weight $\mu = -\lambda_{n-1}$, choose $\nu = \frac{1}{n} \sum_{i=1}^n \varepsilon_i$. Then we get \mathfrak{so}_{2n+1} .



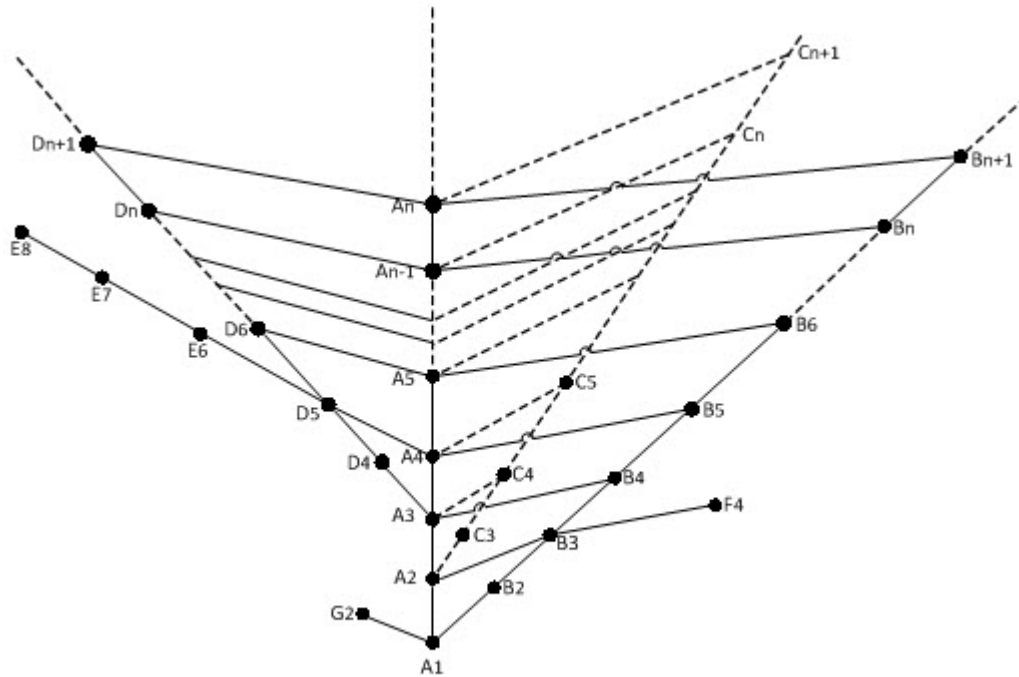
(2) The C_n series. Take $\mathfrak{g} = \mathfrak{sl}_n$ and $\text{sym}^2 V$ the quantum symmetric square of vector representation, with lowest weight $\mu = -2\lambda_{n-1}$, choose $\nu = \frac{2}{n} \sum_{i=1}^n \varepsilon_i$. Then we get \mathfrak{sp}_{2n} .



(3) The D_n series. Take $\mathfrak{g} = \mathfrak{sl}_n$ and $\wedge^2 V$ the second quantum exterior power of vector representation, with lowest weight $\mu = -\lambda_{n-2}$, choose $\nu = \frac{2}{n} \sum_{i=1}^n \varepsilon_i$. Then we get \mathfrak{so}_{2n} .



In order to give the readers an intuitive understanding, based on our constructions in section 3 as well as those in [HH1, HH2, HH3], we draw the tree of quantum groups with nodes $U_q(\mathfrak{g})$'s for all the finite dimensional complex simple Lie algebras \mathfrak{g} , which grow out of the source node A_1 inductively by a series of suitably chosen double-bosonization procedures. This is the Majid's main expectation for his conjecture ([M1, M6]).



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